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Two-level mortar domain decomposition preconditioners for heterogeneous elliptic problems

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Highlights

- We use nonoverlapping domain decomposition for mixed method approximations.
- We propose a two-level preconditioner based on the interfaces between subdomains.
- The coarse preconditioner uses the multiscale mortar domain decomposition method.
- Prolongation is defined uniquely to preserve projection onto normal velocities.
- Numerical tests of highly heterogeneous porous media show efficiency and robustness.

Abstract

We consider a second order elliptic problem with a heterogeneous coefficient, which models, for example, single phase flow through a porous medium. We write this problem in mixed form and approximate it for parallel computation using the multiscale mortar domain decomposition mixed finite element method, which gives rise to a saddle point linear system. We use a relatively fine mortar space, which allows us to enforce continuity of the normal velocity flux, or nearly so in the case of nonmatching meshes. To solve the Schur complement linear system for the mortar unknowns, we propose a two-level preconditioner based on the interfaces between subdomains. The coarse preconditioner also uses the multiscale mortar domain decomposition method, but with instead a very coarse mortar space. We show that the prolongation operator of the coarse mortar to the fine is defined uniquely by the condition that the L^2-projection of a coarse mortar agrees with its projection onto the space of normal velocity fluxes, i.e., no energy is introduced when changing mortar scales. The local smoothing preconditioner is based on block Jacobi, using blocks defined by the interfaces. We use restrictive smoothing domains that are smaller normal to the interfaces, and overlapping in the directions tangential to the interfaces. In the simplest case, the condition number of the preconditioned interface operator is bounded by a multiple of $(\log(1 + H/h))^2$. We show several numerical examples involving strongly heterogeneous porous media to demonstrate the efficiency and robustness of the preconditioner. We see that it is often desirable, and

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sometimes necessary, to use a piecewise linear or higher order coarse mortar space to achieve good convergence for heterogeneous problems.

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1. Introduction

On a domain $\Omega \subset \mathbb{R}^d,\ d = 2$ or 3, we consider the second order elliptic problem

$$\mathbf{u} = -a\nabla p \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{3}$$

wherein ν is the outer unit normal vector to the domain, which is written in mixed form, i.e., as a system of two first order equations plus the boundary condition. The equation arises from minimizing the functional $F(\mathbf{u}, p) = \frac{1}{2}E(\mathbf{u}) + \int_\Omega (\nabla \cdot \mathbf{u} - f)\, p\, dx$, where $E(\mathbf{u}) = \int_\Omega a^{-1}|\mathbf{u}|^2\, dx$ is the energy of the system and p enforces the divergence constraint. We target applications involving flow in porous media [1,2], in which case p is the fluid pressure, $\mathbf{u}$ is the (Darcy) velocity, and the coefficient a is the permeability. The permeability is often highly anisotropic and heterogeneous, varying by many orders of magnitude from point to point. In fact, often the permeability has narrow channels within which the flow is concentrated. These channels are high in permeability and correlated for great distances in some directions but not in others (see, e.g., [3]).

Understanding and predicting fluid flow processes is critical in many subsurface applications, such as CO_2 sequestration, nuclear waste storage, and oil and natural gas production. Furthermore, the flow problem is one of the most time-consuming parts of these simulations. With the development of reservoir characterization methods and geostatistical modeling techniques, the description of reservoir properties can be detailed at multiple scales, from core scales (centimeters) to geological scales (kilometers). A typical reservoir or aquifer is extremely large, and so the geocellular model may have billions of mesh elements. Subsurface processes often last hundreds of years, as in the case of CO_2 migration, or even millions of years for nuclear contaminants. Therefore, we can only simulate these processes using massively parallel supercomputers.

One way of tackling this problem is to reduce its size through upscaling or multiscale techniques [4]. However, the accuracy of the upscaled solution can deteriorate with increasing channel correlation length. Moreover, the flow solution is often coupled to a transport problem, which can magnify errors associated to the flow (see Section 2.2). Our goal is to solve the system on a fine-scale and use multiscale ideas to design effective and robust two-level preconditioners that are suitable for parallel computing when combined with a Krylov accelerator. It is not a new idea to use multiscale ideas to design preconditioners (see, e.g., [5–7]), multigrid methods (e.g., [8,9]), and other iterative procedures (e.g., [10]).

Our approach is to use domain decomposition methods to increase parallelism. In pioneering work, Glowinski and Wheeler [11] defined a nonoverlapping domain decomposition approach to solve the mixed system (1)–(3). We base our work on this method, as modified later to incorporate a general mortar space [12], which became the *multiscale mortar mixed method* [13,14]. This method fully resolves the problem within the subdomains and glues them together with a mortar finite element space. A multiscale solution is obtained when the mortar uses only a few degrees of freedom per interface between subdomains. We use a relatively fine scale mortar space to obtain a fine-scale solution, and a coarse mortar space to define the coarse level preconditioner. The key is to define the extension operator R_0^T from the coarse to the fine mortar space. In fact, we will show that it is uniquely defined by the energy minimizing condition (16) that the L^2-projection of a coarse mortar agree with the projection of its extension, where the projection is onto the space of traces of the normal component of the velocity on the interfaces between subdomains.