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Coexistence of two families of sub-harmonic resonances in a time-delayed nonlinear system at different forcing frequencies

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Two coexisting families of sub-harmonic resonances can be induced at different forcing frequencies in a time-delayed nonlinear system having quadratic nonlinearities. They occur in the region where two stable bifurcating periodic solutions coexist in the corresponding autonomous system following two-to-one resonant Hopf bifurcations of the trivial equilibrium. The forced response is found to demonstrate small- and large-amplitude quasi-periodic motion under the family of sub-harmonic resonances related to Hopf bifurcation frequencies, and large-amplitude periodic and quasi-periodic motion under the family of sub-harmonic resonances associated with the shifted Hopf bifurcation frequencies. The family of sub-harmonic resonances related to Hopf bifurcation frequencies may cease to exist with the loss of the initially established frequency relationship of sub-harmonic resonances when the magnitude of periodic excitation is beyond a certain value. This will lead to a jump phenomenon from small- to large-amplitude quasi-periodic motion. Bifurcation diagrams, time trajectories and frequency spectra are numerically obtained to characterize the sub-harmonic resonances of the time-delayed nonlinear system around the critical point of the resonant Hopf bifurcations.

1. Introduction

Time delays can inherently appear in many engineering systems with feedback control including, magnetic bearing systems [1], manufacturing process [2–6], microelectromechanical systems [7], actively controlled mechanical systems [8], vehicle systems [9,10], and spacecraft [11]. Such systems have been referred to as time-delayed nonlinear systems in order to distinguish them from conventional nonlinear systems without time delay, and concurrently time-delayed nonlinear equations used to represent the corresponding mathematical models [12]. Generally speaking, there are three issues needed to be addressed for time-delayed nonlinear systems, which are (1) determination of the critical values of time delays where the trivial equilibrium will lose its stability, (2) identification of bifurcation behaviours of the trivial equilibrium, and (3) characterization of the forced response resulting from the dynamic interactions between the periodic excitation and stable bifurcating solutions (*SBSs*) generated from Hopf bifurcations. The first two issues can be classified as local stability and bifurcations and are related to the autonomous time-delayed nonlinear systems (no excitation is presented in the system), while the third issue as dynamic interactions and is for the non-autonomous time-delayed nonlinear systems.

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Many researchers have investigated the local stability and bifurcations of the autonomous time-delayed nonlinear systems which can exhibit different behaviour from the conventional nonlinear systems. Specifically, the trivial equilibrium of autonomous time-delayed nonlinear systems may lose its stability via single or double Hopf bifurcations when the characteristic equation has one or two pairs of purely imaginary eigenvalues [13–17]. The postcritical behaviour of the time-delayed nonlinear systems may exhibit stable periodic motion or unstable divergent motion following single Hopf bifurcations. The frequency of stable bifurcating periodic solution is related to the frequency of Hopf bifurcations which is different from the so-called linearized natural frequency of the time-delayed nonlinear systems. An interaction of double Hopf bifurcations may generate non-resonant or resonant Hopf bifurcations at the bifurcation point of co-dimension two, depending on the frequency ratio of two Hopf bifurcations. Resonant Hopf bifurcations can lead to interactions of the *SBSs* which produce more complicated behaviour.

The periodic excitation presented in non-autonomous time-delayed nonlinear systems can dynamically interact with the *SBSs*. The resultant dynamic interactions can induce resonant oscillations in the forced response and are not yet fully explored for time-delayed nonlinear systems. Primary and secondary resonances were studied for the time-delayed nonlinear systems in the neighbourhood of single Hopf bifurcations [18–21]. Additionally, combination resonances such as additive and difference types were found to exist in the neighbourhood of non-resonant Hopf bifurcations [22]. In other words, only one family of resonances was found for the time-delayed nonlinear systems considered in the existing studies. Resonant Hopf bifurcations of the trivial equilibrium can induce more complex dynamic behaviours than single or non-resonant Hopf bifurcations. For example, they can induce the co-existence of two *SBSs* in the autonomous time-delayed nonlinear system [23]. The dynamic interactions between the periodic excitation and either of the *SBSs* can create different families of resonances in the corresponding non-autonomous system at different forcing frequencies (will be briefly discussed in Section 2.3).

This paper will study two families of sub-harmonic resonance response in a time-delayed quadratic nonlinear system. The coexistence of sub-harmonic resonances in a time-delayed nonlinear system can be considered as a new dynamic phenomenon that has not yet been reported in the literature. It has no analogy in one degree-of-freedom conventional nonlinear system with only one external excitation, because the conventional system has only one type of sub-harmonic resonances depending on the order of nonlinear terms and the forcing frequency. The cascades of combined bifurcations (resonances) may be only possible for one degree-of-freedom conventional nonlinear system with two external periodic excitations, as discussed for a Duffing's oscillator with two external periodic forces in Ref. [24]. It should be noted that the combination resonances in classical nonlinear oscillator with two excitations occur when the two forcing frequencies satisfy certain relationships with the system's linearized natural frequency. On the contrary, the combination resonances happen in a timed-delayed nonlinear system of one-degree-of-freedom having only one periodic excitation when the two frequencies of bifurcating periodic solutions satisfy certain relationships with the forcing frequency.

The remainder of this paper is organized as follows. Section 2 briefly discusses two coexisting *SBSs* and different families of primary and secondary resonances. Two families of sub-harmonic resonance response are numerically studied in Sections 3 and 4. Section 5 presents the concluding remarks.

2. Coexistence of the *SBSs* and different families of resonances

This section provides a brief background on the resonant Hopf bifurcations, the coexistence of the *SBSs*, the approximate solutions to the family of sub-harmonic resonance response related to Hopf bifurcation frequencies, and then introduces the different families of resonances.

2.1. The time-delayed nonlinear system and resonant Hopf bifurcations

A mass-damper-spring system with feedback control, as shown in Fig. 1, can be regarded as a simple model for describing the oscillations of time-delayed nonlinear mechanical systems. The nonlinear spring has a linear-plus-quadratic characteristic stiffness. The quadratic nonlinearities are commonplace in engineering systems and may come from nonlinear elastic forces, nonlinear stress-strain relationships, and geometrical deformations such as initial curvatures and buckled states of flexible structures. The vertical tire force of a vehicle is also a quadratic nonlinear function of the vertical tire deflection in the first approximation. The time delay is considered as delayed reactions of actuators or purposely introduced in the control loop. Periodic excitations considered may come from the eccentricity of unbalance and inertia effects in mechanical systems.

The dimensionless equation of motion determining the oscillations of the mass can be given by:

$$\ddot{x} + \mu\dot{x} + \omega^2 x + \alpha x^2 = e\cos(\Omega t) + px(t-\tau) + q\dot{x}(t-\tau) + k_1 x^2(t-\tau), \tag{1}$$

where x is the displacement, μ is the damping coefficient, ω is the so-called linearized natural frequency, α is the coefficient of the quadratic term, e and Ω denote the amplitude and frequency of the periodic excitation, and an over-dot represents the differentiation with respect to time t. The parameters, τ, p, q, and k_1 stand for the time delay, the linear and quadratic feedback gains, respectively. The procedure for deriving Eq. (1) can be found in Ref. [25].