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Novel bursting patterns in a Van der pol-Duffing oscillator with slow varying external force

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ABSTRACT

In this paper, we investigate the emergence of bursting dynamics with complex waveforms and their relation to periodic behavior in typical Van der pol-Duffing equation with fifth order polynomial stiffness nonlinearity, when the external force changes slowly with the variation of time. We exploit bifurcation characteristics of the fast subsystem using the slowly changing periodic excitation as a bifurcation parameter to show how the bursting oscillations are created in this model. We also identify that some regimes of bursting patterns are related to codimension two bifurcation type over a wide range of parameters. A subsequent two-parameter continuation reveals a transition in the bursting behavior from fold/fold hysteresis cycle to sup-Hopf/sup-Hopf or limit point cycle/sub-Hopf bursting type. Furthermore, the effects of external forcing item on bursting oscillations are investigated. For instance, the time interval between two adjacent spikes of bursting oscillations is dependent on the forcing frequency. Some numerical simulations are included to illustrate the validity of our study.

1. Introduction

In the past several decades, the study of bursting oscillations have received great attention. Bursting oscillations are waveforms that consist of alternating small and large amplitude excursions. Such oscillations have been reported in both experiments and models of systems from chemistry, biochemistry, physics and neuroscience [1–5]. In fact, bursting oscillations are often observed to occur in signal transduction processes and associated with many biological phenomena, such as oscillatory enzyme responses [6], insulin production in the β-cell [7] and veratridine in rat injured sciatic nerves [8,9].

For a fast-slow dynamic, its dynamical behaviors can be described by a singularly perturbed system with two time scales of the following form [10–12],

$$\begin{aligned} \dot{x} &= f(x,u) \quad \text{(Fast Spiking)} \\ \dot{u} &= \varepsilon g(x,u) \quad \text{(Slow Modulation)} \end{aligned} \tag{1}$$

where $\varepsilon \ll 1$ represents the ratio of time scales between spiking and modulation. Vector $x \in \mathbb{R}^n$ models the dynamics of a relatively fast changing processes, while vector $u \in \mathbb{R}^k$ describes the relatively slowly changing quantity that modulates x. A standard method of analysis of birsting oscillations, introduced by Rinzel et al. [13–15], is to set $\varepsilon = 0$ and consider the

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fast and the slow subsystems separately, which is known as the dissection of neuronal bursting since it allows us to study the fast subsystem x and treat u as a vector of slowly changing bifurcation parameters.

Van der Pol-Duffing equation, which is known to describe many important oscillating phenomena in nonlinear engineering systems, has become one of the commonest examples in nonlinear oscillation texts and research articles [16–20]. Many efforts have been made to find its approximate solutions or to construct simple maps that qualitatively describe the important features of its dynamics. In the classical Van der Pol-Duffing oscillator, it was shown that the oscillations occur when a stable equilibrium undergoes the singularity induced bifurcation, which in turn corresponds to the occurrence of supercritical Hopf bifurcations in the singularly perturbed models [21–24].

In our recent papers, using Rinzel's method, we investigated dynamical behaviors of some multiple coupled systems with form of Eq. (1) and found some novel bursting patterns [25–28]. Here we will present an analysis of bursting oscillations driven by external forcing for Van der Pol-Duffing equation. Bursting oscillations to be studied in this paper are different from the usual fast-slow bursters since the trajectories are tumbling around the multiple equilibria and driven by external forcing. The purpose of this work is two fold. First, we study bursting oscillations induced by external forcing as well as how the external forcing modulates the bursting dynamics. Secondly, some important aspects of Rinzel's method are highlighted in the multi-stable oscillator. In particular, we argue the particular constellation of a subcritical Hopf bifurcation together with a limit point cycle bifurcation.

The rest of this paper is organized as follows. In the next section, the forced equation and its fast-slow analysis are presented. In Section 3, in order to investigate bursting oscillations, we consider the external force $f\cos(\omega t)$ as a slow variable and study its codimension one and two bifurcation on the controlled Van der Pol-Duffing equation as well as its influence. In Section 4, two strategies are developed to reveal the dynamical mechanisms of bursting oscillations. In Section 5, we focus on the effects of forcing amplitude and forcing frequency on bursting oscillations. Finally, Section 6 concludes the paper.

2. Mathematical model and its slow manifold

In this paper, we consider the classical Van der Pol-Duffing equation with fifth order polynomial stiffness nonlinearity and periodic excitation, which is often written in the form

$$\ddot{x} + \delta(x^2 - 1)\dot{x} + \alpha_1 x - \alpha_2 x^3 + \alpha_3 x^5 = f\cos(\omega t), \quad (2)$$

where δ ($\delta > 0$) can be regarded as dissipation or damping, α_1 is linear stiffness parameter, α_2 and α_3 mean nonlinear stiffness, and parameters f ($f > 0$) is the forcing amplitude and ω ($\omega > 0$) is the forcing frequency. This model exhibits many qualitatively different phenomena and is considered as one of the most intensely studied nonlinear systems and has served as a basic model in physics, electronics, biology, neurology and so on.

Since we are interested in the case when $fcos(\omega t)$ changes slowly, we assume $0 < \omega \ll 1$ here which implies the excited frequency ω is far small from the natural frequency Ω of the oscillator. The effects of multiple time scales appear, which results in different types of bursting oscillations under different parameters. The oscillatory behaviors of bursting behave in periodic states characterized by a combination of relatively large amplitude (spiking process) and nearly harmonic small amplitude oscillations (quasi-stationary process), conventionally denoted by N^k where N and K correspond to large and small amplitude oscillations respectively.

The external excited oscillator can be considered as the coupling of two autonomous subsystems by regarding $fcos(\omega t)$ as a generalized state variable ρ, the fast subsystem (FS) of which is presented as the following equation

$$\ddot{x} + \delta(x^2 - 1)\dot{x} + \alpha_1 x - \alpha_2 x^3 + \alpha_3 x^5 = \rho, \quad (3)$$

while the slow manifold is written as $\rho = fcos(\omega t)$. It can be checked that $\dot{\rho} = -f\omega sin(\omega t)$, which forms the slow manifold for the far small value of $\omega \ll 1$. In the following, two important bifurcation behaviors of the FS can be discussed. One corresponds to bifurcation from a quiescent state to repetitive spiking mode, while the other will induce the spiking attractor back to the quiescent state.

3. Bifurcation analysis for the fast subsystem

Inspired by the dissection of neuronal bursting [29], we begin our study of the mechanism responsible for bursting oscillations by considering the slow variable ρ as a control parameter and studying its influence on the controlled Van der Pol-Duffing oscillator, which may help to understand bursting oscillations induced by the stability and bifurcation dynamics in the fast subsystem.

3.1. Analysis for Cusp bifurcation

Simple in form as it is, the equilibria of FS can be written in the form $(x, y) \equiv (x, \dot{x}) = (x_0, 0)$, where x_0 is decided by the real roots of the following algebraic equation

$$-\alpha_1 x + \alpha_2 x^3 - \alpha_3 x^5 + \rho = 0. \quad (4)$$