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A priori error estimation for the stochastic perturbation method

Xiang-Yu Wang[a,b], Song Cen[a,d,e], C.F. Li[b,c,*], D.R.J. Owen[b]

[a] *Department of Engineering Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*
[b] *Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Swansea SA2 8PP, UK*
[c] *Energy Safety Research Institute, College of Engineering, Swansea University, Swansea SA2 8PP, UK*
[d] *High Performance Computing Centre, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*
[e] *Key Laboratory of Applied Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*

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Abstract

The perturbation method has been among the most popular stochastic finite element methods due to its simplicity and efficiency. The error estimation for the perturbation method is well established for deterministic problems, but until now there has not been an error estimation developed in the probabilistic context. This paper presents *a priori* error estimation for the perturbation method in solving stochastic partial differential equations. The physical problems investigated here come from linear elasticity of heterogeneous materials, where the material parameters are represented by stochastic fields. After applying the finite element discretization to the physical problem, a stochastic linear algebraic equation system is formed with a random matrix on the left hand side. Such systems have been efficiently solved by using the stochastic perturbation approach, without knowing how accurate/inaccurate the perturbation solution is. In this paper, we propose *a priori* error estimation to directly link the error of the solution vector with the variation of the source stochastic field. A group of examples are presented to demonstrate the effectiveness of the proposed error estimation.

Keywords: Error estimation; Stochastic finite element methods; The perturbation method

1. Introduction

Over the past few decades, Stochastic Finite Element Methods (SFEM) have been developed for numerical analysis of uncertainty propagation in various engineering applications, in which the random factors can arise from the geometry, the material properties or the boundary conditions. Unlike the conventional finite element method which has a well established theoretical framework, the SFEM has many different formulations, and these include the Monte Carlo method [1–4], the perturbation method [5–7], the Neumann expansion method [8,9], the polynomial chaos expansion method [10–13], and the joint diagonalization method [14–16], among others. Despite these developments, little work has been done to address the error estimation of SFEMs. This work focuses on the error estimation of the

* Correspondence to: College of Engineering, Swansea University, Swansea SA2 8PP, UK.
E-mail address: c.f.li@swansea.ac.uk (C.F. Li).

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stochastic perturbation method. The basic idea and the development of the stochastic perturbation method is briefly recapped below. We also summarize the Monte Carlo method, which is taken as the reference for the error estimation.

The Monte Carlo method [1–4,17–19] generates samples for the source random variables involved, and corresponding to each sample a deterministic problem is solved to obtain the sample response, after which the statistical evaluation is conducted on the whole set of sample responses. Among all SFEM approaches, the Monte Carlo approach is the simplest and also the most versatile method. However, its computational cost is usually high and in many cases, becomes unmanageable due to the limited computing resources available.

The basic idea of the stochastic perturbation method [5–7,20–22] is to expand the stochastic quantities by a Taylor expansion, equate the items with the same order and calculate the responses. Due to the simplicity and efficiency of the perturbation method, it has been widely used in statics, thermodynamics, structural optimization, dynamics, etc. Chang et al. [23] investigated the stochastic responses of geometrically nonlinear beams and frames, which have uncertain material and geometric properties and are subjected to random excitations both spatially and temporally. Zhang et al. [24] analyzed the effects of random material properties on the elastic stability of structural members and frames. Lee et al. [25] presented an optimal design method to include the structural uncertainty. Kaminski et al. [26] applied second order perturbation to homogenization problems. Lei et al. [27] presented an approach to analyze structures with stochastic parameters under random excitation. Doltsinis et al. [28] discussed the robust design of structures with stochastic parameters. Onkar et al. [29] proposed a method for the buckling analysis of both homogeneous and laminated plates with random material properties. Cavdar et al. [30] demonstrated an example of predicting the performance of structural systems with uncertainties. Fan et al. [31] studied the robust optimization of large-scale space structures under thermal loads. Guedri et al. [32] investigated uncertainty propagation for viscoelastic systems. Lepage et al. [33] applied the perturbation stochastic finite element method to the homogenization of polycrystalline materials. Kaminski et al. [34] employed the perturbation-based stochastic finite elements to analyze composite materials with stochastic interface defects, and later they [35] applied the generalized stochastic perturbation method to thermal stresses and deformation analysis of spatial steel structures exposed to fire. In order to achieve better accuracy in the stochastic perturbation approach, Kaminski et al. [36,37] applied higher-order Taylor expansion and tested the procedure on simple problems with analytical solutions. Falsone et al. [38–40] developed a perturbation-like procedure for static and dynamic analysis of FE discretized structures with uncertainties, which overcomes the drawbacks related to the standard perturbation approach.

Despite the wide use in various applications [25,26,28–35], the perturbation method does not have a rigorous error estimation. Instead, the solution errors are usually examined by comparing with the Monte Carlo method. Purely based on experience, some papers [6,22,36,41] suggested an applicable variation range of ten percent. Without rigorous proof, it is generally agreed that the perturbation method has very good efficiency and accuracy for small random fluctuations, but the solution accuracy will decrease dramatically as the variation gets larger.

For error estimation, Oden et al. [42] provided a general error estimation for the error of the mathematical model, the error of numerical approximation, and the error of random fluctuation of the parameters, in which the variation of the parameters was approximated with the first order perturbation. *A posteriori* error estimation was derived. Babuska et al. [43] studied the issues involved in the solution of linear ordinary differential equations with random parameters, and proved the existence of a rigorous error estimation for the perturbation method. However, the proposed error estimation contains unknown constants which cannot be estimated either *a priori* or *a posteriori*. A similar work for linear elliptic problems with random parameters was presented in [44], where the existence of the error estimation was proved without providing *a priori* or *a posteriori* evaluation.

This work addresses the *a priori* error estimation for the stochastic partial differential equations commonly encountered in stochastic finite element methods. The relative error is directly evaluated from the variation of the source stochastic field. Section 2 explains the problem description and the general setup of *a priori* error estimation. In Sections 3 *and* 4, some preliminary theorems are first introduced, after which the detailed *a priori* estimation is derived. A series of numerical examples are given in Section 5 to demonstrate the effectiveness of the error estimation. Conclusions are given in Section 6.

2. Problem description

Using a stochastic finite element approach to solve problems associated with heterogeneous materials, the random material property is usually described by a stochastic field which is modeled with the spectral representation [45,46],