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A bi-partitioned iterative algorithm for solving linear systems arising from incompressible flow problems

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Highlights

- Iterative algorithm to solve linear system in stabilized finite elements for fluids.
- Uses Schur complement to separately solve for velocity and pressure.
- Order of magnitude improvement compared to GMRES while maintaining stability.

Abstract

A novel iterative algorithm, called the bi-partitioned method, is introduced for efficiently solving the system of linear equations that arises from the stabilized finite element formulation of the Navier–Stokes equations. The bi-partitioned algorithm is a Krylov subspace method designed for a matrix with separated momentum and continuity blocks. This structure allows for formation of the Schur complement to separately solve for the velocity and pressure unknowns. Hence, the bi-partitioned algorithm can also be applied to problems with similar matrix structure, involving the Schur complement. Two separate spaces are constructed iteratively from the velocity and pressure solution candidates and optimally combined to produce the final solution. The bi-partitioned algorithm calculates the final solution to a given tolerance, regardless of the approximation made in construction of the Schur complement. The proposed algorithm is analyzed and compared to the generalized minimal residual (GMRES) algorithm using two incompressible-flow and one fluid–structure-interaction example, exhibiting up to an order of magnitude improvement is simulation cost while maintaining excellent stability.

Keywords: Schur complement; Finite elements; Fluid–structure interaction; Outflow boundary conditions; Stabilized formulation; GMRES

1. Introduction

Finite element (FE) methods for fluid mechanics are increasingly adopted to simulate a wide range of engineering and scientific problems ranging from aeronautics to biomechanics. More complex problems, involving multiple

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physical systems and scales, are arising with ever increasing computational cost. For example, multiscale simulations approaches have been adopted to model the circulatory system, and fluid–structure interaction (FSI) has been adopted to study wind turbines and ventricular assist devices [1–7]. Additionally, performing optimization and uncertainty quantification requires conducting numerous CFD simulations in a robust and accurate way [8,9]. In these scenarios, computational cost of simulations can become critically important due to the relative scarcity of computational resources, and the need to obtain results in an acceptable time frame. Hence, reducing the cost of CFD simulations remains a pressing problem that continues to be an active area of research (see [10–18] and references therein).

Solving systems of linear equations generally constitutes the majority of the total cost of CFD solvers that use implicit time integration schemes. In this context, direct methods are generally avoided due to their unfavorable scalability with system size, and unreasonably high cost for engineering problems [10]. Iterative methods, are generally the preferred choice due to their favorable scalability characteristics and more modest memory requirements. The generalized minimal residual method (GMRES) remains the most widely adopted iterative scheme for FE flow simulations, though other methods have at times shown good performance in specialized applications [15,11,19–23]. The purpose of this study is to introduce a specialized numerically stable and cost efficient algorithm as an alternative to the GMRES method, to solve systems of linear equations arising from the FE discretization of the incompressible Navier–Stokes equations.

The Schur complement is generally constructed by decomposing the left-hand-side (LHS) matrix into four blocks, with one well-conditioned diagonal block, as common in the FE discretization of fluid mechanics problems [13,14]. In the context of iterative methods, the well-conditioned diagonal block allows one to efficiently solve the associated linear system iteratively or to reasonably approximate it with a diagonal matrix. In either case, the accuracy of the entire algorithm, and hence the solution, will be controlled by this approximation. In the present study we introduce a novel algorithm, that we call the bi-partitioned (BIPN) method, to efficiently circumvent this limitation. In contrast to the conventional approaches that are based on a single solution space, the BIPN method constructs the final solution from an optimal combination of two separate spaces with higher degrees of freedom, hence achieving any desired accuracy with a faster convergence rate. Although this algorithm is presented in the context of FE methods for CFD, it can also be extended to other applications involving the Schur complement.

This paper is organized as follows: First, a standard stabilized FE formulation for fluid mechanics and its time integration are presented for completeness and to ensure reproducibility of the results. Second, the BIPN algorithm is introduced and presented in detail. Third, three examples, a straight channel, a geometrically complex model of the aorta, and a well known benchmark FSI problem, are computed using the BIPN algorithm and compared with the conventional GMRES technique. Fourth, we discuss the results and draw conclusions.

2. Methods

In this section, we first review the FE discretization of the Navier–Stokes equations and the underlying scheme that produces the system of linear equations (Section 2.1). We then propose our BIPN algorithm for solving that system in Section 2.2. Note that although we describe a specific numerical scheme in Section 2.1, the BIPN method is applicable to other discretization schemes and stabilized formulations that produce a system of linear equations with similar structure. In what follows, roman subscripts are used to denote variable names and italic superscripts are used as indices.

2.1. Finite element formulation

For an incompressible and Newtonian fluid, the weak formulation of the Navier–Stokes equations is as follows. Find $\boldsymbol{u} \in \boldsymbol{\mathcal{S}}$ and $p \in \mathcal{P}$, such that for all $\boldsymbol{w} \in \boldsymbol{\mathcal{W}}$ and $q \in \mathcal{Q}$

$$
\begin{aligned}
&B_{\mathrm{G}}(\boldsymbol{w}, q; \boldsymbol{u}, p) = F_{\mathrm{G}}(\boldsymbol{w}, q),\\
&B_{\mathrm{G}} = \int_{\Omega} \left[\rho \boldsymbol{w} \cdot (\dot{\boldsymbol{u}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla \boldsymbol{w} : (-p\boldsymbol{I} + \mu \nabla^{\mathrm{s}} \boldsymbol{u}) + q \nabla \cdot \boldsymbol{u}\right] \mathrm{d}\Omega - \int_{\Gamma_{\mathrm{h}}} \boldsymbol{w} \cdot \boldsymbol{h}_{\mathrm{c}} \mathrm{d}\Gamma,\\
&F_{\mathrm{G}} = \int_{\Omega} \rho \boldsymbol{w} \cdot \boldsymbol{f} \mathrm{d}\Omega + \int_{\Gamma_{\mathrm{h}}} \boldsymbol{w} \cdot \boldsymbol{h}_{\mathrm{u}} \mathrm{d}\Gamma,
\end{aligned}
\tag{1}
$$