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Control-based continuation: Bifurcation and stability analysis for physical experiments

David A.W. Barton

Department of Engineering Mathematics, University of Bristol, United Kingdom

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ABSTRACT

Control-based continuation is technique for tracking the solutions and bifurcations of nonlinear experiments. The idea is to apply the method of numerical continuation to a feedback-controlled physical experiment such that the control becomes non-invasive. Since in an experiment it is not (generally) possible to set the state of the system directly, the control target becomes a proxy for the state. Control-based continuation enables the systematic investigation of the bifurcation structure of a physical system, much like if it was numerical model. However, stability information (and hence bifurcation detection and classification) is not readily available due to the presence of stabilising feedback control. This paper uses a periodic auto-regressive model with exogenous inputs (ARX) to approximate the time-varying linearisation of the experiment around a particular periodic orbit, thus providing the missing stability information. This method is demonstrated using a physical nonlinear tuned mass damper.

1. Introduction

Control-based continuation is a systematic method for performing bifurcation studies on physical experiments. Based on modern feedback control schemes it enables dynamical phenomena to be detected and tracked as system parameters are varied in a similar manner to how nonlinear numerical models can be investigated using numerical continuation. Control-based continuation was originally developed as an extension to Pyragas' time-delayed feedback control [1–3] to make it more robust and suitable for parameter studies [4], though its current implementation contains no elements of time-delayed feedback.

The use of feedback control for the investigation of nonlinear systems is not new; in addition to time-delayed feedback, methods such as OGY control [5] have previously been employed to provide non-invasive control to stabilise unstable orbits and investigate dynamical phenomena. Indeed, previous authors have gone as far as to implement such control schemes within parameter continuation studies to numerically simulate particular experiments such as atomic force microscopes [6] or reaction kinetics [7]. Control-based continuation goes beyond these particular methods to allow the use of almost any feedback control scheme and, as such, it is a general purpose tool applicable to a wide range of physical experiments.

Control-based continuation has been successfully applied to a range of experiments including a parametrically excited pendulum [8], nonlinear energy harvesters [9,10] and a bilinear oscillator [11,12]. In each case, periodic orbits have been tracked through instabilities such as saddle-node bifurcations (folds) thus revealing a great deal of dynamical information

E-mail address: david.barton@bristol.ac.uk

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about the system in question, including the location of a codimension-2 cusp bifurcation in one case [13]. As well as bifurcations, other dynamic features such as backbone curves can also be tracked with control-based continuation [14].

Although the basic scheme for control-based continuation is well established (an overview is provided in Section 2), it lacks many of the features of standard numerical continuation schemes such as bifurcation detection. Only saddle-node bifurcations (folds) can be detected readily and that is because they are geometric features in the solution surface. Bifurcations such as period-doubling bifurcations are not geometric features and can go undetected due to the stabilising affect of the feedback controller. Similarly, the inclusion of the feedback controller means that methods for calculating eigenvalues/Floquet multipliers and basins of attraction from experiments such as [15–17] are not helpful since they indicate the stability of the closed-loop system rather than the open-loop system.

In this paper we consider only periodically forced systems and hence study periodic orbits, though there is no reason that the methods developed should not be applicable to autonomous systems as well. In Section 3, we propose a method for calculating the stability (the Floquet multipliers and associated stable and unstable eigendirections) based on the estimation of a local linearisation around a stabilised periodic orbit. We demonstrate the effectiveness of this method in Sections 4 and 5 by applying it to a (physical) nonlinear mass–spring–damper system where the nonlinearity is geometric in nature—the springs are mounted perpendicular to the direction of motion.

2. Control-based continuation

Numerical continuation is a path following method used to track solution branches as parameters of the system in question are varied. In a nonlinear system, these solution branches can encounter bifurcations at particular parameter values which results in a qualitative change in the dynamics of the system. Numerical continuation enables these bifurcations to be detected and tracked in turn. It is typically applied to differential equation models but it can be used more widely, for example on finite element models.

At a basic level, numerical continuation tracks the solutions of an arbitrary nonlinear function, a *zero problem* given by

$$f(x, \lambda) = 0, \quad f: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m \tag{1}$$

where x is the system state and λ is the system parameter(s). A common example of this is tracking the equilibria of an ordinary differential equation with respect to a single parameter. In this case $n = m$ and $p = 1$ in Eq. (1), that is, $f=0$ defines a one-dimensional curve. Alternatively the function f can arise from the discretisation of a periodic orbit. Numerical continuation works in a predictor-corrector fashion; at each step a new solution $\tilde{x}$ is predicted from previously determined solutions and then the solution is corrected using a nonlinear root finder applied to the function f (typically a Newton iteration). The use of a nonlinear root finder means that the stability or instability of solutions is unimportant. In certain circumstances (for example, near a fold or saddle-node bifurcation) the function f must be augmented with an additional equation—the pseudo-arclength equation—which enables the numerical continuation scheme to track solution curves that double back on themselves. In these circumstances, without the pseudo-arclength equation the correction step for a fixed set of parameter values λ will fail since no solution exists. For extensive information and guidance on numerical continuation see the textbooks [18,19]. Numerical software is also readily available in the form of CoCo [20] and AUTO [21] amongst others.

Control-based continuation is a means for defining a zero-problem based on the outputs of a physical experiment, thus enabling numerical continuation to be applied directly without the need for a mathematical model. To do this there are two key challenges to overcome: (1) In general, it is not possible to set the state x of the physical system and so it is not possible to evaluate f at arbitrary points. (2) The physical system must remain around a stable operating point while the experiment is running. While a numerical model going unstable might prove to be a mild annoyance, a physical system going unstable can prove dangerous.

In order to overcome these challenges, a feedback controller is used to stabilise the system and the control target (or reference signal) acts as a proxy for the system state. The feedback controller takes the form

$$u(t) = g(x^*(t) - x(t)) \tag{2}$$

where $x^*(t)$ is the control target and g is a suitable control law such as proportional-derivative (PD) control (as used in this paper) where

$$u(t) = K_p(x^*(t) - x(t)) + K_d(\dot{x}^*(t) - \dot{x}(t)). \tag{3}$$

For the method outlined in this paper, the choice of control law is at the discretion of the user; any suitable stabilising feedback control scheme can be used. The challenge here is to devise a scheme for embedding the feedback control within the numerical continuation such that the controller becomes non-invasive, that is, the controller does not affect the locations of any invariant sets in the experiment such as equilibria or period orbits. This requirement for non-invasiveness defines the zero problem; a control target must be chosen such that the control action

$$u(t) \equiv 0. \tag{4}$$