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Fundamental aspects of shape optimization in the context of isogeometric analysis

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Highlights

- We provide a framework for shape optimization problems with isogeometric analysis.
- For that, we express shape gradients in isogeometric terms.
- Equivalence of discrete systems from discretize-first and optimize-first is shown.
- We formulate shape optimization with respect to NURBS in the optimize-first ansatz.

Abstract

We develop a mathematical foundation for shape optimization problems under state equation constraints where both state and control are discretized by B-splines or NURBS. In other words, we use isogeometric analysis (IGA) for solving the partial differential equation and a nodal approach to change domains where control points take the place of nodes and where thus a quite general class of functions for representing optimal shapes and their boundaries becomes available. The minimization problem is solved by a gradient descent method where the shape gradient will be defined in isogeometric terms. This gradient is obtained following two schemes, *optimize first–discretize then* and, reversely, *discretize first–optimize then*. We show that for isogeometric analysis, the two schemes yield the same discrete system. Moreover, we also formulate shape optimization with respect to NURBS in the optimize first ansatz which amounts to finding optimal control points and weights simultaneously. Numerical tests illustrate the theory.

Keywords: Isogeometric analysis; Shape optimization; Adjoint approach; Weight optimization; NURBS

1. Introduction

Isogeometric analysis (IGA) combines the fundamental idea of the finite element method (FEM) with spline techniques from computer aided geometric design for a common description of the domain and the Galerkin projection [1]. IGA aims to overcome the bottleneck of converting design-suitable descriptions to FEM-suitable

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models, and it holds particularly great promise in the field of shape optimization where the frequent conversion between geometry description and computational mesh is cumbersome and error-prone.

From a mathematical perspective, a nodal approach to find the optimal shape, i.e. using a piecewise linear interpolation of the domain's boundary, is mostly not desirable because of regularity and well-posedness issues. One possible way out is to parameterize the boundary by B-splines [2,3]; and obviously, the approximation or exact representation of a domain by means of B-splines and NURBS is a better choice than using a space of polygons.

Therefor, the combination of shape optimization with isogeometric analysis seems very favorable because all occurring approximation spaces can be covered by one common description, namely B-splines or NURBS, and there is the benefit of arbitrary regularity of boundary interpolation. However, in IGA not only the boundary is parameterized in this way but also the inside, leading to new options in shape optimization methods. The combination of IGA and shape optimization has already been investigated in a number of papers such as [4,5] with application to electromagnetism and [6–9] with application to solid mechanics and also shells [10].

It is the objective of this paper to introduce a general framework that clarifies certain aspects and sheds new light on the solution of shape optimization problems by means of IGA. In particular, we discuss the two different approaches *discretize first–optimize then* vs. *optimize first–discretize then* with gradient-based shape optimization and show that the order of optimization and discretization commutes for shape optimization in IGA. Though being a common statement in optimal control theory, this equivalence of the two approaches has certain restrictions and specific consequences.

Optimization with partial differential equations as state constraints is an active research area with interconnections to functional analysis and various other fields. Reaching out to a broader audience in the engineering community, our exposition here tries to compromise between a rigorous mathematical treatment and a more informal discussion of the subject that highlights the main ideas and concepts. Throughout the paper, we assume a given cost functional

$$J(u, \Omega, \Gamma) := \int_{\Omega} j_1(u, x)\mathrm{d}x + \int_{\Gamma} j_2(u, s)\mathrm{d}s \tag{1}$$

for domains $\Omega \subset \mathbb{R}^d,\ d = 2, 3$, with moving boundary $\Gamma \subset \partial\Omega$, and a general shape optimization problem

$$\min J(u, \Omega, \Gamma) \quad \text{s.t. } E(u, \Omega) = 0 \tag{2}$$

with a state equation

$$E(u, \Omega) = 0 \tag{3}$$

representing a second order linear elliptic partial differential equation with solution $u := u(\Omega)$.

At an optimal shape Ω^*, formal differentiation of (2) w.r.t. Ω yields the necessary optimality condition

$$\mathrm{d}_{\Omega} J(u^*, \Omega^*, \Gamma^*) = 0. \tag{4}$$

The crucial point in shape calculus is how to define the shape derivative d_{Ω}. More specifically, the main problem here is that domains are sets and as such the space of admissible shapes has no vector space nor topological structure. Hence, adding domains as well as speaking of distances between them makes no sense—let alone making statements about convergence and differentiability. Shape calculus methods such as the method of perturbation of identity, or speed method, overcome these deficiencies by providing both structures. As one of the basic references in this field, we refer to [11], and a rigorous mathematical treatment is provided by [12]. The *discretize first* point of view is treated in [2] whereas the *optimize first* approach in the form of Lagrange multipliers can be found in [13], among others. Except for [9,8], the *discretize first* ansatz is so far used in IGA.

There also are several other angles from which shape gradients can be viewed, for instance [14] from a Riemannian perspective which might be even more natural to the isogeometric setting than perturbation of identity. We also want to point to [15,16] for more shape optimization problems and in particular for topology optimization, which we do not consider in this work.

In the following, we will first introduce IGA in Section 2 and form the space $\mathcal{G}$ for admissible shapes. We claim that shape calculus in IGA is tailored towards these parameterizations in $\mathcal{G}$, but even though, it is just a special case of the method of perturbation of identity. We will review the latter briefly in Section 3.1 before specializing it to shape calculus in IGA in Section 3.2. Section 4 utilizes this theory to show that in IGA *discretize first–optimize then* is the