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A unified approach to strict upper and lower bounds of quantities in linear elasticity based on constitutive relation error estimation

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Highlights

- A unified representation of various quantities is presented in this paper.
- Even pointwise quantities are shown to fit well with the unified representation.
- Dual error estimation is conducted based on the unified representation.
- Two bounding approaches are revisited and optimized for the goal-oriented error estimation.
- Strict upper and lower bounds of displacement integrals, stress integrals and pointwise quantities are acquired by the present unified approach.

Abstract

This paper presents a unified approach to acquiring strict upper and lower bounds of various quantities in linear elasticity. The key ingredient lies in a unified representation of linear quantities including displacement integrals, stress integrals and even pointwise quantities. With the unified representation, dual error analysis can be performed easily which results in a unified approximation and thereby a unified error representation via the primal–dual equivalence theorem. Then, the constitutive relation error (CRE) estimation featured with the ability to provide strict upper bound of global energy norm error is utilized and strict upper and lower bounds of the quantities are obtainable thereafter. Moreover, two extant bounding approaches to goal-oriented error estimation are analyzed and optimized, and as a result, optimal approximation and bounds are obtained. Numerical examples are studied to validate the proposed unified approach.

Keywords: Constitutive relation error (CRE); Unified representation of quantities; Optimized bound; Bounds of pointwise quantities; Goal-oriented error estimation

1. Introduction

Nowadays, finite element (FE) analysis has been conducted widely and routinely in engineering design. On the one hand, it facilitates practical design in a rather effective way. On the other hand, inaccuracy which is harmful for

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robust design often arises inevitably due to existence of errors. These errors are of various sources [1], including errors in the geometry, loadings, behavior of material, discretization and so on, among which the discretization error constitutes the main ingredient. The process to assess and control the global discretization error is often referred to as "*model verification*". Towards this end, a great deal of research work has been carried out in last decades. Three different families of error estimation have been established [2]: the first is the recovery-based error estimation whose representative work belongs to Zienkiewicz and Zhu [3]; the second is the residual-type error estimation, being reflected mainly in Babuška and Rheinboldt's work [4]; the third is the constitutive relation error (CRE) estimation in which the framework was proposed by Ladevèze and his co-workers [5]. Among these families, the CRE estimation has stood out due to its capability of providing strict and computable error bound of the global discretization error.

The CRE estimation inherits the core idea of dual analysis proposed as early as 1964 by Fraeijs de Veubeke [6,7]. In the dual analysis, a kinematically admissible displacement field is demanded, as well as a statically admissible stress field which often involves equilibrium finite elements based on the minimum complementary energy principle [8–10]. The admissible displacement field is naturally available in conventional displacement-based FE analysis; however, when it comes to the admissible stress field, as pointed out by Maunder [11], the cost of the global equilibrium finite element analysis is often prohibitively high. To avoid global computation, constructing the admissible stress field from FE results in a local and efficient way has become the main stream of the CRE estimation [1]. Three main techniques have been proposed to construct the admissible stress field. They are the element equilibration technique (EET) [5,12–14] where equilibrated element tractions are first acquired and the admissible stress field is computed in an element-by-element manner thereafter, the star-patch equilibration technique (SPET) [12] (or termed the *flux/traction-free* technique [15–17]) where the admissible stress field is computed patch-wisely without *a priori* knowledge of equilibrated tractions and the element equilibrium + star-patch technique (EESPT) [12,18]. Each of the three techniques has its relative advantages and disadvantages in terms of effectivity and computational cost (see [12] for an overview). In this paper, the EET is implemented in the CRE estimation due to its relatively inexpensive computational cost.

In fact, the knowledge of strict bounds of global energy norm error does not make direct sense for engineering design. For instance, in structural analysis, it is often the error in estimation of quantities such as internal forces at a beam section and the maximum effective stress that is of concern, rather than the global energy norm error. To evaluate an arbitrary quantity of interest by use of the strict global energy norm error bound, a corresponding *dual* (or called *adjoint*) problem [19–22] is introduced and then the error of the quantity is represented by the mixture of global energy errors in both the *primal* and dual problems. Error estimation of the quantity can proceed thereafter. Various kinds of error estimation have led to various types of goal-oriented error estimations [23–26]. In particular, the one in combination with the CRE estimation has made strict bounds of quantities obtainable with the aid of the strict bounding property of CRE estimation. This excellent property has been exploited in analysis of many problems and consequently, the strict bounds of various quantities of interest have been obtained, such as the stress intensity factors in fracture mechanics [27,28], pointwise quantities of elastic problems [29,30], quantities in viscoelasticity, plasticity and dynamic problems [25,31–35] and so on.

Though great achievements have been made in the CRE-based goal-oriented error estimation, the upper and lower bounds of various quantities are often obtained following a tortuous course. In general, linear quantities can be categorized into three groups: displacement integrals, stress integrals and pointwise quantities. In the current framework, goal-oriented error estimation works well on the first group and thereby, quantities in other two groups have to be transformed into the first group *a priori* which is often termed *extraction* [1]. For stress integrals, usually, the transformation into the first group can be easily fulfilled through integration by parts [36], but in some cases, such as the total flux on complex Dirichlet boundary [37], the extraction may be unavailable. For quantities such as pointwise displacements, pointwise stresses and stress intensity factors at a crack tip, singularities arise during the transformation and special techniques [38,39] have to be implemented to extract these quantities. And yet, the procedure is never easily conducted [40,41]. Obviously, the diversified, tortuous and even intractable treatment of quantities for evaluation of strict bounds has hindered to some extent the development of the goal-oriented error estimation. It would be rather appealing if a unified approach to bounds of an arbitrary quantity is available and automated computation [42] for bounds of an arbitrary quantity becomes achievable. In fact, a unified representation for linear quantities, differing from that for quantities in the first group, has been proposed by Giles and Süli in [20]. In this paper, the unified representation is followed up and further enriched. It is shown that any quantity in the three groups can be easily transformed into a unified form and the corresponding dual problem can be established in an analogous way. Furthermore, within the framework of the CRE estimation, two bounding approaches—*Washizu*'s and *Greenberg*'s approaches [43]