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Stochastic modeling of uncertain mass characteristics in rigid body dynamics

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ABSTRACT

This paper focuses on the formulation, assessment, and application of a modeling strategy of uncertainty on the mass characteristics of rigid bodies, i.e. mass, position of center of mass, and inertia tensor. These characteristics are regrouped into a 4×4 matrix the elements of which are represented as random variables with joint probability density function derived following the maximum entropy framework. This stochastic model is first shown to satisfy all properties expected of the mass and tensor of inertia of rigid bodies. Its usefulness and computational efficiency are next demonstrated on the behavior of a rigid body in pure rotation exhibiting significant uncertainty in mass distribution.

1. Introduction

The mass of a rigid body is often considered to be a parameter of a dynamic model that is easily identifiable and thus requires no further consideration. A similar perspective is also often taken of the two other mass distribution related characteristics of rigid bodies, i.e. position of center of mass and inertia tensor. Even if a direct measurement of these latter characteristics is more challenging than that of the mass, all of them can certainly be determined very accurately from a computer-aided model of the body considered *provided* that the mass distribution and exact shape of the body are well known.

This information is however not always available with the desirable accuracy because of natural variability. For example, the mass distribution of a vehicle is dependent on the number, locations, and weights of the passengers (see [1] for discussion and modeling) which will usually be quite variable. The mass distribution of a rocket during lift-off also exhibits variability owing to the fuel burning process. Finally, unbalance in rotating systems is a manifestation of variability in the mass distribution that, even when small, may have dramatic effects. To study such systems, it is desirable to have a modeling strategy of the variability in the mass characteristics that permits the estimation of the band ("uncertainty band") in which the corresponding predicted results will fall as the mass characteristics are varied.

As discussed in details below, two different types of strategies can be utilized to carry out this modeling. The first one proceeds directly at the detailed level of the mass distribution and possibly shape of the body, e.g. see Refs [2–4]. The second, which is of interest here, focuses directly on the desired characteristics, i.e. mass, position of center of mass, and inertia tensor, see [1,5], with the latter paper involving both rigid body and flexible motions of a structure. In this regard, it must be recognized that the modeling of the mass characteristics of rigid bodies discussed here is not derived from the consideration of the mass matrix associated with the 6 rigid body modes of translations and rotations of a free-free structure. The difference is rooted in the necessity for the mass associated with all translations to be the same and these modes to remain uncoupled as the mass distribution is varied. That is, additional constraints exist when considering rigid bodies which are not present/considered in the context of flexible structures for

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which a broader uncertainty modeling literature is available (e.g. see [6]).

In this light, the focus of the present paper is on the development, assessment, and application of a stochastic modeling approach of the mass, position of center of mass, and inertia of rigid bodies within the framework described in [5]. This effort is similarly in intent as the work carried out in [1] but it will proceed differently, focusing on a 4×4 positive definite matrix which conveniently regroups mass, inertia, and center of gravity position for a straightforward maximum entropy stochastic modeling. An important aspect of the present effort will be to prove that these simulated mass characteristics will effectively satisfy all expected properties for rigid bodies, which are briefly reviewed first. The usefulness and computational efficiency of the proposed methodology will be demonstrated on a rigid body dynamic problem in which there is significant uncertainty in the mass characteristics.

2. Mass characteristics of a rigid body – key properties

Excluding gravitational attraction terms, the mass distribution of a rigid body affects its dynamics through only 3 particular characteristics: its mass, the position of its center of mass, and its tensor of inertia with respect to one of its points. As is well known, the specification of these quantities provides only minimal information on the mass distribution of the rigid body. Yet, these characteristics are not completely independent of each other. Indeed, consider the vector

$$\underline{W}_{OP} = [1 \quad x \quad y \quad z]^T \tag{1}$$

where the superscript ^{T} denotes the operation of vector/matrix transposition and x, y, and z are the coordinates of a point P of the body as measured in a particular frame of reference $\underline{i}$, $\underline{j}$, $\underline{k}$ from the reference point O belonging to the body. Introduce next the matrix $\underline{\underline{\mu}}_O$ (double underbars are used throughout to denote matrices/second order tensors) as

$$\underline{\underline{\mu}}_O = \int_{Body} \underline{W}_{OP} \, \underline{W}_{OP}^T \, dm \tag{2}$$

where the integral is carried out over the body and dm is the corresponding element of mass. This matrix combines all three mass characteristics mentioned above as it can be expressed as

$$\underline{\underline{\mu}}_O = \begin{bmatrix} m & \underline{u}_{OG}^T \\ \underline{u}_{OG} & \underline{\underline{J}}_O \end{bmatrix} \tag{3}$$

where m is the body mass, $\underline{u}_{OG}$ the product of the mass by the position vector of the center of mass G from O, and $\underline{\underline{J}}_O$ is the Euler tensor of the body with respect to point O. The corresponding tensor of inertia $\underline{\underline{I}}_O$ is then obtained as

$$\underline{\underline{I}}_O = tr[\underline{\underline{J}}_O] \, \underline{\underline{I}}_3 - \underline{\underline{J}}_O \tag{4}$$

where $tr[\underline{\underline{U}}]$ denotes the trace of an arbitrary matrix $\underline{\underline{U}}$ and $\underline{\underline{I}}_3$ is the 3×3 identity matrix.

By construction, the matrix $\underline{\underline{\mu}}_O$ defined in Eq. (2) is symmetric and positive and this property implies that the tensor of inertia $\underline{\underline{I}}_O$ is positive definite and furthermore that its counterpart at the center of mass $\underline{\underline{I}}_G$ is also positive definite as required (e.g. see [5] for proofs). Thus, for a particular point O and a particular set of axes $\underline{i}$, $\underline{j}$, $\underline{k}$, the positive definiteness and symmetry of the corresponding matrix $\underline{\underline{\mu}}_O$ are the key properties.

A second property characterizing this matrix results from the consideration of different sets of axes. Specifically, introduce a second frame of reference $\underline{i}'$, $\underline{j}'$, $\underline{k}'$ in which the point P of Eq. (1) has coordinates x', y', z' related to x, y, z by the coordinate transformation matrix $\underline{\underline{A}}$, i.e. with

$$[x' \quad y' \quad z']^T = \underline{\underline{A}} \quad [x \quad y \quad z]^T \tag{5}$$

Then, the corresponding vector $\underline{W}'_{OP}$ is

$$\underline{W}'_{OP} = [1 \quad x' \quad y' \quad z']^T = \begin{bmatrix} 1 & \underline{0}^T \\ \underline{0} & \underline{\underline{A}} \end{bmatrix} [1 \quad x \quad y \quad z]^T = \underline{\underline{B}} \quad \underline{W}_{OP} \tag{6}$$

and thus the matrix $\underline{\underline{\mu}}'_O$ defined as in Eq. (2) with the vector $\underline{W}'_{OP}$ can be expressed in terms of $\underline{\underline{\mu}}_O$ as

$$\underline{\underline{\mu}}'_O = \underline{\underline{B}} \, \underline{\underline{\mu}}_O \underline{\underline{B}}^T \tag{7}$$

Note finally that the matrix $\underline{\underline{\mu}}_A$ obtained with respect to a point A can be expressed in terms of its counterpart for point O in the same frame of reference, $\underline{\underline{\mu}}_O$, by noticing that

$$\underline{W}_{AP} = \underline{W}_{OP} + \begin{bmatrix} 0 \\ \underline{r}_{AO} \end{bmatrix} \tag{8}$$

where $\underline{r}_{AO}$ denotes the position vector of point O from A.

Introducing this relation in Eq. (2) expressed for point A leads to the relation

$$\underline{\underline{\mu}}_A = \underline{\underline{\mu}}_O + \begin{bmatrix} 0 & m \, \underline{r}_{AO} \\ m \, \underline{r}_{AO}^T & (\underline{r}_{AO} \, \underline{u}_{OG}^T + \underline{u}_{OG} \, \underline{r}_{AO}^T + m \, \underline{r}_{AO} \, \underline{r}_{AO}^T) \end{bmatrix} \tag{9}$$