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Robust error bounds for finite element approximation of reaction–diffusion problems with non-constant reaction coefficient in arbitrary space dimension

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Abstract

We present a fully computable a posteriori error estimator for piecewise linear finite element approximations of reaction–diffusion problems with mixed boundary conditions and piecewise constant reaction coefficient formulated in arbitrary dimension. The estimator provides a guaranteed upper bound on the energy norm of the error and it is robust for all values of the reaction coefficient, including the singularly perturbed case. The approach is based on robustly equilibrated boundary flux functions of Ainsworth and Oden (2000) and on subsequent robust and explicit flux reconstruction. This paper simplifies and extends the applicability of the previous result of Ainsworth and Vejchodský (2011) in three aspects: (i) arbitrary dimension, (ii) mixed boundary conditions, and (iii) non-constant reaction coefficient. It is the first robust upper bound on the error with these properties. An auxiliary result that is of independent interest is the derivation of new explicit constants for two types of trace inequalities on simplices.

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Keywords: Finite element analysis; Robust a posteriori error estimate; Singularly perturbed problems; Boundary layers; Mixed boundary conditions

1. Introduction

Consider a linear reaction–diffusion problem in a domain $\Omega \subset \mathbb{R}^d$ with mixed boundary conditions:

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \Gamma_{\mathrm{D}}; \qquad \partial u/\partial \boldsymbol{n} = g_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{1}$$

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where $\boldsymbol{n}$ stands for the unit outward normal vector to the boundary $\partial\Omega$. The dimension $d \geq 2$ is chosen arbitrarily. For simplicity we assume Ω to be a polytope. The portions Γ_{D} and Γ_{N} of the boundary $\partial\Omega$ are open, disjoint, and satisfy $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} = \partial\Omega$. The reaction coefficient $\kappa \geq 0$ is considered to be piecewise constant. In order to guarantee unique solvability of (1), we consider $\kappa > 0$ in a subdomain of Ω of a positive measure or a positive measure of Γ_{D}. We use the finite element method to approximate the exact solution u by a piecewise affine function u_h with respect to a simplicial partition $\mathcal{T}_h$ of Ω.

In this paper we derive a computable a posteriori error estimate based on robust flux equilibration and explicit flux reconstruction. This error estimate η provides a guaranteed and fully computable upper bound on the energy norm of the error $|||u - u_h|||$ and it is robust with respect to both κ and the mesh-size h.

A posteriori error estimates are useful for adaptive algorithms, where they play two roles. Firstly, they indicate where the computational mesh should be refined or coarsened. Secondly, they provide quantitative information about the size of the error for reliable stopping criterion. Unfortunately, many existing estimators do not provide actual numerical bounds that can be used as a stopping criterion.

Adaptive algorithms are convergent [1] provided the error estimates are *locally efficient* and *reliable*. If η_K stand for local error indicators on elements $K \in \mathcal{T}_h$ and $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$ is the global error estimator, then the indicators η_K are said to be *locally efficient* if there exists a constant $c > 0$ such that

$$c\eta_K \leq |||u - u_h|||_{\widetilde{K}},$$

where $|||u - u_h|||_{\widetilde{K}}$ stands for the energy norm restricted to a patch $\widetilde{K}$ of elements consisting of K and neighbouring elements sharing at least one vertex with K. Similarly, the error estimator η is *reliable* if there exists a constant $C > 0$ such that

$$|||u - u_h||| \leq C\eta.$$

The error estimate η is *robust* if the constants c and C are independent of κ and mesh-size h. The error estimate η is a *guaranteed upper bound* if $|||u - u_h||| \leq \eta$, i.e. the reliability constant C is equal to one. Finally, the error bound η is *fully computable* if it can be evaluated in terms of the approximation u_h and given data without the need for generic (unknown) constants.

A robust, reliable, locally efficient explicit a posteriori error estimate for problem (1) was first derived by Verfürth in [2,3]. This estimate, however, does not provide guaranteed upper bound on the error. An estimator which does provide an upper bound along with robust local efficiency was obtained by Ainsworth and Babuška in [4], but this upper bound depends on an exact solution of a Neumann problem and as noted in [4] is not fully computable. Subsequently in [5] we were able to develop fully computable error bounds in the two dimensional setting by a complementarity technique combined with robustly equilibrated fluxes and explicit flux reconstruction. Here, we develop a simpler flux reconstruction that is suitable for any dimension $d \geq 2$ and is applicable to the case of piecewise constant coefficient κ including the situation where κ can be very large in some parts of the domain and vanishingly small in others. Furthermore, we extend the previous results by considering nonhomogeneous Neumann boundary conditions. In order to achieve these goals, we develop some new techniques and tools for the analysis that are of wider applicability than the problem addressed here.

The question of robust a posteriori error estimates for singularly perturbed problems is studied by other authors as well. In [6], an error estimate that is robust with respect to anisotropic meshes is obtained, but unfortunately does not provide guaranteed upper bound on the error. A robust, locally efficient and fully computable guaranteed upper bound was obtained in [7] for the finite volume method and $d = 2$ and 3. Recently, a robust estimator for the error in the maximum norm was obtained in [8] for the case $d = 1$.

The basic idea behind our work can be traced back to the method of the hypercircle [9] and later to [10–12]. This approach has been adopted by Repin [13] and his group for a wide class of problems in conjunction with the solution of a global minimization problem to compute the error bound. We avoid any global computations and instead develop local algorithms for guaranteed and fully computable error bounds based on *flux equilibration* [4,14–20] etc. In the present work we will utilize the robust flux equilibration from [4].

The rest of the paper is organized as follows. Section 2 defines the finite element approximation and corresponding assumptions. The core of the paper lies in Section 3, where we present new trace inequalities on simplices, and develop two new flux reconstructions both of which are used to derive the a posteriori error and our main result. Finally, Section 4 provides an illustrative numerical example and Section 5 draws the conclusions.