Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 281 (2014) 220–249
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A novel computational formulation for nearly incompressible and nearly inextensible finite hyperelasticity

Adam Zdunek[a,*], Waldemar Rachowicz[b], Thomas Eriksson[c]

[a] *Swedish Defence Research Agency FOI, Aeronautics and Systems Integration, SE-164 90 Stockholm, Sweden*
[b] *Institute for Computer Modelling, Section of Applied Mathematics, Cracow University of Technology, Pl 31-155 Cracow, Poland*
[c] *Department of Engineering Science, Institute of Biomedical Engineering, University of Oxford, UK*

Received 25 February 2013; received in revised form 4 August 2014; accepted 13 August 2014
Available online 23 August 2014

Abstract

A novel approach to computational almost inextensible transversely isotropic and nearly incompressible finite hyperelastic fibre mechanics is introduced. It relies on using an equivalent generalised right Cauchy–Green stretch tensor which is volume preserving and simply-stretch free in the limit of incompressibility and inextensibility. In other words, its third and fourth principal invariants become trivial. Otherwise it represents volume change and fibre stretch with the aid of point-wise equivalent auxiliary measures in the continuous case. The generalised kinematics implies the usual orthogonal spherical–deviatoric decomposition of the stresses. The deviatoric stresses are further orthogonally decomposed into axial fibre- and ground substance-stresses. The novel approach implies that the deviatoric ground substance stresses are trivial in the fibre direction as opposed to the current standard formulation. The approach is also able to represent exact inextensible fibres which is a problem recently addressed in the literature using an additive volumetric–isochoric decoupled strain energy density function, relying on volume preserving stretch. The formulation is corroborated by a couple of numerical examples using a preliminary finite element setting. The basis for the implementation is provided.

Keywords: Anisotropic; Hyperelasticity; Hu–Washizu; Incompressible; Inextensible; Mixed formulation

1. Introduction

This work presents a computational framework for the phenomenological theory of transversely isotropic finite hyperelastic materials. The approach is especially aimed for materials which are nearly incompressible and may become nearly inextensible at finite strains. Typical applications can be found in soft tissue biomechanics and in the mechanics of fibre-reinforced rubber-like materials. The boundary value problems in these applications have to be solved by numerical methods like the finite element method (FEM). The provided framework extends the current displacement, pressure and dilatation formulation for isotropic nearly incompressible finite hyperelastic solids to the anisotropic case.

* Corresponding author. Tel.: +46 855503226, +46 739472819 (Mob.); fax: +46 8 555 0 30 62.
E-mail addresses: zka@foi.se (A. Zdunek), wrachowicz@pk.edu.pl (W. Rachowicz), thomas@tseeriksson.se (T. Eriksson).

http://dx.doi.org/10.1016/j.cma.2014.08.008
0045-7825/

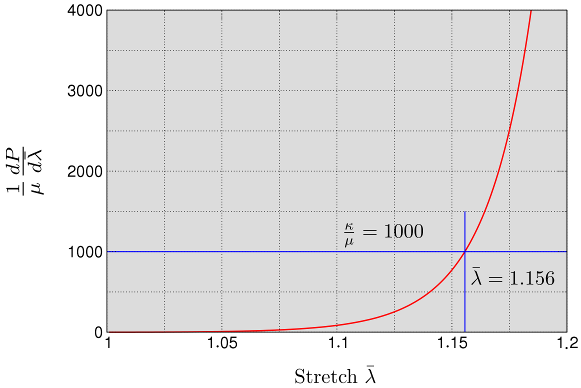

Fig. 1. Normalised fibre extensional stiffness as a function of isochoric fibre stretch $\bar{\lambda}$. Material parameters for *L*eft *A*nterior *D*escending (LAD) coronary human artery adventitia represented by a HGO material model [24] with parameters $\mu = 2.7$ (kPa), $k_1 = 5.1$ (kPa) and $k_2 = 15.4$ (–) according to [33]. The normalised fibre extensional stiffness exceeds the normalised bulk stiffness for $\bar{\lambda} > 1.156$.

Fig. 1 shows the extensional collagen fibre stiffness for a human artery normalised by the artery ground substance shear stiffness μ as a function of the volume preserving part of the fibre stretch $\bar{\lambda}$. A representative ratio between the artery ground substance bulk stiffness κ and its shear stiffness is $\kappa/\mu = 1000$. Fig. 1 shows that the normalised extensional fibre stiffness reaches this value at the realistic fibre strain of 16%. These stiffness ratios will penalise volume change and fibre extension, respectively. They call for special computational methods in finite precision arithmetic. The problem of competing imposition of the corresponding volume- and extensibility-constraint is neither recognised nor previously addressed to our knowledge. An appropriate setting of the problem is given here.

The augmenting volumetric and extensibility constraints coming into play in the limiting case imply stability requirements on the FEM formulation, in the Ladyzhenskaya, Babuška, Brezzi (LBB) sense [1]. The LBB criterion is also known as the *inf–sup condition*, Bathe [2]. Stable discrete formulations for the incompressible case alone exists and are today extensively used. The construction and validation of a stable FEM formulation for the finite hyperelastic (near) inextensible case is on the other hand, to our knowledge, essentially an open problem. It needs an appropriate strong formulation in the first place. A proper setting is provided here (Section 4). A preliminary higher order FEM formulation is used to corroborate the formulation (Sections 7 and 8).

Existing efficient formulations for nearly incompressible isotropic materials rely on the split into isochoric and volumetric deformation measures due to Flory [3]. The third invariant of the isochoric Cauchy–Green stretch is trivial. By definition it is incapable to measure volume changes. The material (constitutive) assumptions used in computational practice are adapted to fit the isochoric–volumetric decoupled form. It is noteworthy(!) that this adaptation is also used for strongly anisotropic materials. The split has become standard in finite element implementations for near incompressibility. Among early contributions that appeared independently are: Simo, Taylor and Pister [4] and Simo and Lubliner [5], Zdunek and Bercovier [6] and Sussman and Bathe [7]. Our starting point is the mixed three-field (displacement, dilatation and pressure) Hu–Washizu formulation developed by Simo and Taylor [8].

The founding developments of the continuum theory of finite deformations of elastic materials reinforced by cords or fibres were contributed by Adkins and Rivlin and are described by Green and Adkins [9]. The approach followed here is the resulting phenomenological theory proposed by Spencer [10,11]. Spencer developed a description where the fibres are characterised by a unit vector field that defines the fibre direction and which deforms with the material. Spencer's approach has close connection to the theory of anisotropic tensor representations involving the use of so-called structural tensors initiated by Boehler [12] and later developed by Zheng [13]. Recently the theory has been applied extensively to soft tissue materials, see for example Holzapfel and Ogden [14].

It is a common practice to use the displacement, dilatation and pressure formulation [8] also for transversely isotropic biological materials with exponentially stiffening fibres, see for example Weiss et al. [15], Holzapfel [16], Gasser et al. [17], Mortier et al. [18], Crane et al. [19], Boerboom et al. [20], Dal et al. [21] and Göktepe [22]. In other words, the near inextensibility of the fibres is not addressed currently in general.

This work provides an appropriate setting for the combined nearly incompressible and nearly inextensible case (Section 3). A generalised Cauchy–Green stretch measure is constructed (Section 2). In the limit of exact incompressibility and exact simple inextensibility its third and fourth principal invariants are trivial. Normally the third