Signal Processing 144 (2018) 29–35

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

An improved RIP-based performance guarantee for sparse signal recovery via simultaneous orthogonal matching pursuit

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ARTICLE INFO

Article history:
Received 17 April 2017
Revised 1 September 2017
Accepted 27 September 2017
Available online 28 September 2017

Keywords:
Compressed sensing
Simultaneous orthogonal matching pursuit
Multiple measurement vectors
Greedy algorithm

ABSTRACT

Recently, based on restricted isometry property (RIP), some sufficient conditions for exact support recovery with simultaneous orthogonal matching pursuit (SOMP) algorithm have been proposed when measurement matrices are different. In this paper, in the noiseless case, one sufficient condition for exact support recovery with SOMP is presented to improve the existing results. By using a counter example, in the noiseless case, an open problem presented in (Xu *et. al*, 2015) is solved.

1. Introduction

The multiple measurement vectors (MMV) problem aims to recover, from sets of compressed measurements, unknown sparse matrices with nonzero entries restricted to a subset of rows. Such problems arise in neuromagnetic imaging [1], source localization [2], and equalization of sparse-communication channels [3]. The aim of the MMV problem is to recover the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ from the observations

$$\mathbf{y}_\ell = \mathbf{A}\mathbf{x}_\ell, \qquad \ell = 1, 2, \cdots, L, \tag{1}$$

where $\Pi := \{1,\ 2,\ \cdots,\ L\}$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$. We define the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ as $\text{supp}(\{\mathbf{x}_\ell\}_{\ell\in\Pi}) = \bigcup_{\ell\in\Pi}\{i \mid \mathbf{x}_\ell(i) \neq 0\}$, where $\mathbf{x}_\ell(i)$ denotes the ith entry of $\mathbf{x}_\ell$ [4].

There have been many studies on the MMV problem associated with (1). In terms of theoretical guarantees, J. Chen et al. [5,6] analyzed the worst-case performance of MMV problem. R. Gribonval et al. [7,8] provided the average-case analysis for MMV. J. Gai et al. [9] presented a high-performance recovery method for MMV. In [10], simultaneous orthogonal matching pursuit (SOMP) was proposed. In [11], several exact recovery criteria ensuring that SOMP identifies the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ have been presented. J. Determe *et al.* [12] provided a novel lower bound for each iteration of SOMP. J. Determe *et al.* [13] presented a theoretical analysis of SOMP operating in the presence of Gaussian additive noise.

The works mentioned above discussed MMV model when the measurement matrices ($\{\mathbf{A}_\ell\}_{\ell\in\Pi}$) for each sparse signal are identical. The authors in [14,15] investigated the case of different $\mathbf{A}_\ell \in \mathbb{R}^{m\times n}$ that is common in the practical applications. For example, to reduce the cost at the nodes implementing compressed sensing (CS) operation, it is advisable to generate $\mathbf{A}_\ell$ independently without communication among them [16–18]. Recently, under the case of different $\mathbf{A}_\ell$, in the noisy case, Xu *et. al* [18] researched the MMV problem using SOMP.

One of the most commonly known tool is the restricted isometry property (RIP, [20]). A matrix $\mathbf{A}$ satisfies RIP of the order K if

$$(1-\delta)\|\mathbf{h}\|_2^2 \le \|\mathbf{A}\mathbf{h}\|_2^2 \le (1+\delta)\|\mathbf{h}\|_2^2 \tag{2}$$

for all K-sparse vector $\mathbf{h}$ with the restricted isometry constant (RIC) $\delta_K \in (0,\ 1)$.

In this study, we mainly recover the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ from the following problem by SOMP in the noiseless case.

$$\mathbf{y}_\ell = \mathbf{A}_\ell\mathbf{x}_\ell, \quad \forall\ \ell \in \Pi. \tag{3}$$

Section 2 describes SOMP and related quantities. In Section 3, we show that if $\mathbf{A}_\ell$ ($\ell \in \Pi$) satisfies the RIP of order $K+1$ with

$$\delta_{K+1} < \frac{1}{\sqrt{LK+1}}, \tag{4}$$

then SOMP can recover the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ in K iterations. Our result (4) is unrelated to the value of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$. Furthermore, in the noiseless case, for any given positive integer K and any $\frac{1}{\sqrt{K+1}} \le \delta < 1$, there always exists a matrix $\mathbf{A}$ satisfying the RIP with $\delta_{K+1} = \delta$

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https://doi.org/10.1016/j.sigpro.2017.09.027

for which SOMP fails to recover the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ in K iterations. We conclude the paper in Section 4. Most of the technicalities are reported in Section 5 to simplify the presentation in the core of the paper.

Now we give some notations. Let $\Omega:=\{1, 2, \cdots, n\}$ and $\Pi:=\{1, 2, \cdots, L\}$. Scalars are written as lowercase letters, e.g., d. We denote vectors by boldface lowercase letters, e.g., $\mathbf{x}$, and matrices as boldface uppercase letters, e.g., $\mathbf{Z}$. For a vector ensemble $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$, $\mathbf{x}_\ell(i)$ means the ith entry of $\mathbf{x}_\ell$. $\{\mathbf{A}_\ell\}_{\ell\in\Pi}$ denotes a matrix ensemble. $\mathbf{A}_{\ell,\Gamma}$ is the matrix containing the columns of $\mathbf{A}_\ell$ indexed by Γ. Similarly, $\mathbf{x}_{\ell,\Gamma}$ is the vector containing the entries of $\mathbf{x}_\ell$ indexed by Γ. Let $\mathcal{P}_{\ell,\Gamma}=\mathbf{A}_{\ell,\Gamma}\mathbf{A}_{\ell,\Gamma}^\dagger$ and $\mathcal{P}_{\ell,\Gamma}^\perp=\mathbf{I}-\mathcal{P}_{\ell,\Gamma}$ denote the orthogonal projection operator onto the column space of $\mathbf{A}_{\ell,\Gamma}$ and its orthogonal complement, respectively. Let $\mathbf{Z}'$ denote the transpose of $\mathbf{Z}\in\mathbb{R}^{n\times L}$. $\mathbf{z}_\ell$ is the ℓth column of $\mathbf{Z}$ and $\mathbf{z}_\ell(i)$ also denotes (i,ℓ)-entry of $\mathbf{Z}$. $\|\mathbf{z}_\ell\|_1=\sum_{i=1}^n|\mathbf{z}_\ell(i)|$ and $\|\mathbf{z}_\ell\|_2=\sqrt{\sum_{i=1}^n(\mathbf{z}_\ell(i))^2}$. $\mathbf{Z}_\Gamma$ is the matrix containing the columns of $\mathbf{Z}$ indexed by Γ. The cardinality of a finite set Γ is denoted by $|\Gamma|$. The support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ is denoted by $\mathrm{supp}(\{\mathbf{x}_\ell\}_{\ell\in\Pi})$, where $\mathrm{supp}(\{\mathbf{x}_\ell\}_{\ell\in\Pi})=\bigcup_{\ell\in\Pi}\{i\mid\mathbf{x}_\ell(i)\neq 0\}$.

2. Problem formulation

Assume that $\Gamma:=\mathrm{supp}(\{\mathbf{x}_\ell\}_{\ell\in\Pi})$ and $|\Gamma|=K$. The SOMP algorithm [18] is listed in Algorithm 1.

Algorithm 1 The SOMP algorithm [18].

Input: $\{\mathbf{y}_\ell\}_{\ell\in\Pi}$, $\{\mathbf{A}_\ell\}_{\ell\in\Pi}$, and K
Initialization: $\mathbf{r}_\ell^0=\mathbf{y}_\ell$ $(\ell\in\Pi)$, $\Lambda^0=\emptyset$, and $k=0$
1: **Repeat** until "stopping criterion" is met
2: $k\leftarrow k+1$
- Match step: $\mathbf{z}_\ell^k=\mathbf{A}_\ell'\mathbf{r}_\ell^{k-1}$ for $\forall\ \ell\in\Pi$
- Identification step: $i^k=\arg\max_i\sum_{\ell=1}^L|\mathbf{z}_\ell^k(i)|$. If multiple maxima exist, choose only one arbitrarily.
- Merge step: $\Lambda^k=\Lambda^{k-1}\bigcup\{i^k\}$
- Update step:
  $\mathbf{x}_\ell^k=\arg\min_{\mathbf{u}:\mathrm{supp}(\mathbf{u})\subseteq\Lambda^k}\|\mathbf{y}_\ell-\mathbf{A}_\ell\mathbf{u}\|_2$ for $\forall\ \ell\in\Pi$
  $\mathbf{r}_\ell^k=\mathbf{y}_\ell-\mathbf{A}_\ell\mathbf{x}_\ell^k$ for $\forall\ell\in\Pi$

3: **End Repeat**
Output: $\{\mathbf{x}_\ell^k\}_{\ell\in\Pi}$ and the recovered support Λ^k

In the $(k+1)$th $(0\le k<K)$ iteration, we begin with the estimated support Λ^k at iteration k. The discussion below demonstrates the generation of Λ^{k+1} [19].

In the update step of Algorithm 1, we need to solve a least square problem, for $\ell\in\Pi$, one has

$$\mathbf{r}_\ell^k=\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{y}_\ell\overset{(a)}{=}\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{A}_\ell\mathbf{x}_\ell\overset{(b)}{=}\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{A}_{\ell,\Gamma\setminus\Lambda^k}\mathbf{x}_{\ell,\Gamma\setminus\Lambda^k}, \tag{5}$$

where (a) follows from (3), (b) follows from $\Gamma=\mathrm{supp}(\{\mathbf{x}_\ell\}_{\ell\in\Pi})$ and the definition of $\mathcal{P}_{\ell,\Lambda^k}^\perp$. When $k=0$, $\Lambda^k=\Lambda^0=\emptyset$. Then the orthogonal projector becomes an identity matrix.

In the matching step, we have

$$\begin{aligned}\mathbf{z}_\ell^{k+1}&=\mathbf{A}_\ell'\mathbf{r}_\ell^k=\mathbf{A}_\ell'\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{y}_\ell=\mathbf{A}_\ell'(\mathcal{P}_{\ell,\Lambda^k}^\perp)'\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{y}_\ell\\&=\mathbf{A}_\ell'(\mathcal{P}_{\ell,\Lambda^k}^\perp)'\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{A}_{\ell,\Gamma\setminus\Lambda^k}\mathbf{x}_{\ell,\Gamma\setminus\Lambda^k}.\end{aligned} \tag{6}$$

So, in the identification step of Algorithm 1, according to [18], one has

$$\sum_{\ell=1}^L|\mathbf{z}_\ell^{k+1}(i)|=0,\qquad\forall\ i\in\Lambda^k. \tag{7}$$

Therefore, one has [18]

$$\arg\max_i\sum_{\ell=1}^L|\mathbf{z}_\ell^{k+1}(i)|\notin\Lambda^k,\quad|\Lambda^k|=k, \tag{8}$$

i.e., SOMP will choose a different index from $\Omega:=\{1, 2, \cdots, n\}$ in each iteration.

3. RIP analysis of SOMP

In this section, to improve the existing results and solve the open problem presented in [18], in the noiseless case, we will present one sufficient condition for exactly recovering the support of $\{\mathbf{x}_\ell\}_{\ell\in\Pi}$ with SOMP in K iterations.

3.1. Preliminary

Let Λ^k denote the estimated support at iteration k. For simplicity, denote

$$\mathbf{Z}^{k+1}=[\underbrace{\mathbf{A}_1'\mathbf{r}_1^k}_{\mathbf{z}_1^{k+1}}\ \ \underbrace{\mathbf{A}_2'\mathbf{r}_2^k}_{\mathbf{z}_2^{k+1}}\ \ \cdots\ \ \underbrace{\mathbf{A}_{L-1}'\mathbf{r}_{L-1}^k}_{\mathbf{z}_{L-1}^{k+1}}\ \ \underbrace{\mathbf{A}_L'\mathbf{r}_L^k}_{\mathbf{z}_L^{k+1}}]\in\mathbb{R}^{n\times L} \tag{9}$$

for $0\le k<K$. $\mathbf{z}_\ell^{k+1}$ is the ℓth column of $\mathbf{Z}^{k+1}$. In other words, $\mathbf{z}_\ell^{k+1}$ gathers the product of the transpose of the ℓth measurement matrix $\mathbf{A}_\ell$ and the ℓth residual $\mathbf{r}_\ell^k$ at iteration k. In that sense, each row of $\mathbf{Z}^{k+1}$ corresponds to a particular support index and will determine whether that particular index will be picked or not.

Let

$$\mathbf{X}^{k+1}=[\mathbf{x}_{1,\Gamma\setminus\Lambda^k}\ \ \mathbf{x}_{2,\Gamma\setminus\Lambda^k}\ \ \cdots\ \ \mathbf{x}_{L,\Gamma\setminus\Lambda^k}]\in\mathbb{R}^{(K-k)\times L}, \tag{10}$$

and

$$\mathbf{B}^{k+1}=[\underbrace{\mathcal{P}_{1,\Lambda^k}^\perp\mathbf{A}_1\mathbf{x}_1}_{\mathbf{b}_1^{k+1}}\ \ \underbrace{\mathcal{P}_{2,\Lambda^k}^\perp\mathbf{A}_2\mathbf{x}_2}_{\mathbf{b}_2^{k+1}}\ \ \cdots\ \ \underbrace{\mathcal{P}_{L,\Lambda^k}^\perp\mathbf{A}_L\mathbf{x}_L}_{\mathbf{b}_L^{k+1}}]\in\mathbb{R}^{m\times L} \tag{11}$$

for $0\le k<K$. $\mathbf{b}_\ell^{k+1}=\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{A}_\ell\mathbf{x}_\ell=\mathcal{P}_{\ell,\Lambda^k}^\perp\mathbf{A}_{\ell,\Gamma\setminus\Lambda^k}\mathbf{x}_{\ell,\Gamma\setminus\Lambda^k}$ is the ℓth column of $\mathbf{B}^{k+1}$ for $\ell\in\Pi$. In that sense, $\mathbf{b}_\ell^{k+1}$ is the residual at iteration k corresponding to the ℓth measurement vector.

The following lemmas are useful in our analysis.

Lemma 1. *[21] If* $\mathbf{A}$ *satisfies the RIP of orders* k_1 *and* k_2 *with* $k_1\le k_2$, *then* $\delta_{k_1}\le\delta_{k_2}$.

Lemma 2. *[22, 23] Suppose that* $\mathrm{supp}(\mathbf{x})=S$. *Let set* S *satisfy* $|S\bigcap\Lambda|\ge 1$. *If* $\mathbf{A}$ *satisfies the RIP of order* $|S\bigcup\Lambda|$ *with constant* $\delta_{|S\bigcup\Lambda|}$, *then*

$$\begin{aligned}(1-\delta_{|S\bigcup\Lambda|})\|\mathbf{x}_{S\setminus\Lambda}\|_2^2\le\|\mathcal{P}_\Lambda^\perp\mathbf{A}\mathbf{x}\|_2^2&=\|\mathcal{P}_\Lambda^\perp\mathbf{A}_{S\setminus\Lambda}\mathbf{x}_{S\setminus\Lambda}\|_2^2\\&\le(1+\delta_{|S\bigcup\Lambda|})\|\mathbf{x}_{S\setminus\Lambda}\|_2^2.\end{aligned}$$

Lemma 3. *As defined in (9), (10) and (11), we have*

$$\|(\mathbf{Z}^{k+1})'_{\Gamma\setminus\Lambda^k}\|_F\ge\frac{1}{\|\mathbf{X}^{k+1}\|_F}\|\mathbf{B}^{k+1}\|_F^2, \tag{12}$$

where $(\mathbf{Z}^{k+1})'_{\Gamma\setminus\Lambda^k}$ $(0\le k<K)$ *is the matrix containing the columns of* $(\mathbf{Z}^{k+1})'$ *indexed by* $\Gamma\setminus\Lambda^k$, *i.e., the rows of* $\mathbf{Z}^{k+1}$ *indexed by* $\Gamma\setminus\Lambda^k$.

The proof of Lemma 3 is given in Section 5.1.

Lemma 4. *To simplify the notation, for given* $j\in\Gamma^c$, *we define*

$$\alpha_{k+1}:=-\frac{\sqrt{L(K-k)+1}-1}{\sqrt{L(K-k)}} \tag{13}$$

and $\mathbf{H}^{k+1}\in\mathbb{R}^{n\times L}$ *with*

$$\mathbf{h}_\ell^{k+1}=\alpha_{k+1}\|\mathbf{X}^{k+1}\|_F\frac{\mathbf{z}_\ell^{k+1}(j)}{\sqrt{\sum_{\ell=1}^L(\mathbf{z}_\ell^{k+1}(j))^2}}\mathbf{e}_j \tag{14}$$