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Frequency-selective Vandermonde decomposition of Toeplitz matrices with applications

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ABSTRACT

The classical result of Vandermonde decomposition of positive semidefinite Toeplitz matrices, which dates back to the early twentieth century, forms the basis of modern subspace and recent atomic norm methods for frequency estimation. In this paper, we study the Vandermonde decomposition in which the frequencies are restricted to lie in a given interval, referred to as frequency-selective Vandermonde decomposition. The existence and uniqueness of the decomposition are studied under explicit conditions on the Toeplitz matrix. The new result is connected by duality to the positive real lemma for trigonometric polynomials nonnegative on the same frequency interval. Its applications in the theory of moments and line spectral estimation are illustrated. In particular, it provides a solution to the truncated trigonometric *K*-moment problem. It is used to derive a primal semidefinite program formulation of the frequency-selective atomic norm in which the frequencies are known *a priori* to lie in certain frequency bands. Numerical examples are also provided.

1. Introduction

A classical result discovered by Carathéodory and Fejér in 1911 [1] states that, if an $N \times N$ Hermitian Toeplitz matrix $\boldsymbol{T}$ is positive semidefinite (PSD) and has rank $r \le N$, then it can be factorized as

$$\boldsymbol{T} = \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^H, \tag{1}$$

where $\boldsymbol{P}$ is an $r \times r$ positive definite diagonal matrix and $\boldsymbol{A}$ is an $N \times r$ Vandermonde matrix whose columns are discrete sinusoidal waves with distinct frequencies. Moreover, such a decomposition is unique if $r < N$. This Vandermonde decomposition result has become important for information and signal processing since the 1970s when it was rediscovered by Pisarenko and used for frequency estimation by interpreting the Toeplitz matrix $\boldsymbol{T}$ as the data covariance matrix. The Vandermonde decomposition in (1) is therefore also referred to as the Carathéodory–Fejér–Pisarenko decomposition. As a result of this rediscovery, a class of methods have been developed for frequency estimation based on the signal subspace of a data covariance estimate, known as the subspace-based methods. Prominent examples are multiple signal classification (MUSIC), estimation of parameters by rotational invariant techniques (ESPRIT) and various variants of them (see the review in [2]). Besides, this decomposition result is important in moment theory, operator theory and system theory [3,4]. As an example, it can be applied to give a solution to the truncated trigonometric moment problem (a.k.a. the moment problem on the unit circle given a finite moment sequence) [5].

In the past few years, a new class of methods for frequency estimation have been devised, namely the gridless sparse methods (see the review in [6]), in which the Vandermonde decomposition is evoked and plays an important role. It is well-known that sparse methods for frequency estimation developed in the past two decades exploit the signal sparsity, which arises naturally from the fact that the number of frequencies is small, and attempt to find, among all candidates consistent with the observed data, the solution consisting of the smallest number of frequencies. Since frequency estimation is a highly nonlinear problem and to overcome such nonlinearity, gridding in the continuous frequency domain used to be a standard ingredient of early sparse methods, which transforms approximately the original nonlinear continuous parameter estimation problem as a problem of sparse signal recovery from a linear system of equations (see, e.g., [7,8]). The newly developed gridless sparse methods completely avoid gridding, work directly in the continuous domain, and have strong theoretical guarantees. These methods have been developed based on

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the atomic norm [9–13]—a continuous analogue of the ℓ_1 norm used in the early sparse methods—and covariance fitting [14]. A main difficulty of applying these gridless sparse methods underlies in how to solve the nonlinearity problem, which makes the resulting optimization problems nonconvex with respect to the unknown frequencies. To do so, the key is to apply the Vandermonde decomposition of Toeplitz matrices to cast these optimization problems as semidefinite programs (SDP), in which the frequencies are encoded in a PSD Toeplitz matrix, as $\boldsymbol{T}$ in (1). Once the SDP is solved, the frequencies are finally retrieved from the Vandermonde decomposition of the solved Toeplitz matrix. Note that the Vandermonde decomposition result has also been generalized to high dimensions and used for multidimensional frequency estimation [15].

Notice that the frequencies in the Vandermonde decomposition in (1) may take any value in the normalized band [0, 1] (or the unit circle), in which 0 and 1 are identified. This paper is motivated by various practical applications in which the (normalized) frequencies can be known *a priori* to lie in certain frequency bands. For example, when a signal is oversampled by a factor, the frequencies will lie in a band narrowed by the same factor. Due to the path loss effect, the maximum value of the range/delay, which can be interpreted as a frequency parameter, of a detectable aircraft can be estimated in advance. Similarly, the maximum Doppler frequency can be obtained if the aircraft's characteristic speed can be known. In underwater channel estimation, the frequency parameters of interest can reside in a known small interval [16]. Similar prior knowledge might also be available given weather observations [17]. Therefore, it would be interesting to exploit such prior knowledge in gridless sparse methods for frequency estimation, and by doing so, the estimation accuracy is expected to improve.

The important role of the Vandermonde decomposition in gridless sparse methods encourages us to incorporate the prior interval knowledge into the decomposition. In other words, we ask the following question: *Can the frequencies in the Vandermonde decomposition of the Toeplitz matrix* $\boldsymbol{T}$*, as in (1), be restricted to lie in a given interval* $\mathcal{I} \subset [0, 1]$*, instead of the entire domain* [0, 1]*, under explicit conditions on* $\boldsymbol{T}$*?* In fact, we also want the conditions to be convex due to our interest in optimization problems. The resulting decomposition is referred to as frequency-selective (FS) Vandermonde decomposition. The question asked above is challenging since, by (1), $\boldsymbol{T}$ is a highly nonlinear function of the frequencies and it is unclear how to link $\boldsymbol{T}$ to a frequency interval $\mathcal{I}$.

It is interesting to note that similar questions have been investigated in a class of moment problems known as truncated K-moment problems, a.k.a. truncated moment problems on a semialgebraic set K, instead of on an entire domain [18]. When K is in the real or the complex domain, solutions to these problems have been successfully obtained [19,20]. To the best of our knowledge, however, the problem is still open when K is defined on the unit circle [0, 1], which is known as the truncated trigonometric K-moment problem. In this paper, we show that the study of the FS Vandermonde decomposition can provide a solution to this open problem.

In this paper, an affirmative answer is provided to the question asked above. Concretely, it is shown that a PSD Toeplitz matrix $\boldsymbol{T}$ admits an FS Vandermonde decomposition on a given interval if and only if $\boldsymbol{T}$ satisfies another linear matrix inequality (LMI). Interestingly, this FS Vandermonde decomposition result is linked by duality to the positive real lemma (PRL) for trigonometric polynomials [21]. The usefulness of the new result is also demonstrated. In the theory of moments, it provides a solution to the truncated trigonometric K-moment problem. For frequency estimation with prior interval knowledge, it is used to derive a primal SDP formulation for the atomic norm exploiting the prior knowledge. Numerical examples are also provided.

1.1. Related work

This paper extends our conference paper [22] in which the FS Vandermonde decomposition of Toeplitz matrices was studied. In addition to this, we show in this paper the connection between the FS Vandermonde decomposition and the PRL for trigonometric polynomials. Its applications to the moment theory and frequency estimation are also studied in more detail.

The problem of frequency estimation with restriction on the frequency band was studied in [23–25]. In [23], an FS atomic norm formulation (or constrained atomic norm in the language of [23]) was proposed and a dual SDP formulation was presented by applying the theory of positive trigonometric polynomials. In contrast to this, we show in this paper that a primal SDP formulation of the FS atomic norm can be obtained by applying the new FS Vandermonde decomposition. In [24], the interval prior was interpreted as a prior distribution of the frequencies and a weighted atomic norm approach was then devised that is an approximate but faster implementation of the FS atomic norm. Although the paper [25] does not provide or imply the FS Vandermonde decomposition result, it obtained independently a primal SDP formulation of the FS atomic norm based on a different technique.

The paper [26] studied the super-resolution problem on semialgebraic sets in the real domain and provided an SDP formulation of the resulting atomic norm. To do so, the key is to apply the moment theory on semialgebraic sets in the real domain (a.k.a. the truncated K-moment problem in the real domain). In contrast to this, we provide a first solution to the truncated trigonometric K-moment problem and then apply this result to study super-resolution on semi-algebraic sets on the unit circle.

1.2. Notations

Notations used in this paper are as follows. $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\mathbb{T} := [0, 1]$ denotes the unit circle, in which 0 and 1 are identified. Boldface letters are reserved for vectors and matrices. $|\cdot|$ denotes the amplitude of a scalar or the cardinality of a set. $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the ℓ_1, ℓ_2 and Frobenius norms respectively. $\boldsymbol{A}^T$ and $\boldsymbol{A}^H$ are the matrix transpose and conjugate transpose of $\boldsymbol{A}$ respectively. rank($\boldsymbol{A}$) denotes the rank and tr($\boldsymbol{A}$) is the trace. For PSD matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \geq \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is PSD. $\Re$ and $\Im$ return the real and the imaginary parts of a complex argument respectively.

A Hermitian trigonometric polynomial of degree one is defined as:

$$g(z) = r_1 z^{-1} + r_0 + r_{-1} z, \quad r_{-1} = \bar{r}_1, \quad r_0 \in \mathbb{R}, \tag{2}$$

where z is a complex argument and $\bar{\cdot}$ denotes the complex conjugate operator. When $\boldsymbol{z}$ is on the unit circle, i.e., when $z = e^{i2\pi f}$, $f \in \mathbb{T}$, we write without ambiguity $g(f) := g(e^{i2\pi f})$. It follows that

$$g(f) = r_1 e^{-i2\pi f} + r_0 + \bar{r}_1 e^{i2\pi f} = r_0 + 2\Re\left\{r_1 e^{-i2\pi f}\right\}, \tag{3}$$

and $g(f)$ is real on $\mathbb{T}$.

An $N \times N$ Toeplitz matrix $\boldsymbol{T} := \boldsymbol{T}(\boldsymbol{t}) := \boldsymbol{T}(N, \boldsymbol{t})$ is formed by using a complex sequence $\boldsymbol{t} = \left[t_j\right]$, $j = 1 - N, \ldots, N - 1$ and defined by $T_{mn} = t_{n-m}$, $0 \leq m, n \leq N - 1$. Given $\boldsymbol{t}$ and a degree-1 trigonometric polynomial g as defined in (2), an $(N - 1) \times (N - 1)$ Toeplitz matrix $\boldsymbol{T}_g := \boldsymbol{T}_g(\boldsymbol{t}) := \boldsymbol{T}_g(N, \boldsymbol{t})$ is defined by

$$[T_g]_{mn} = r_1 t_{n-m+1} + r_0 t_{n-m} + r_{-1} t_{n-m-1}, \tag{4}$$

$0 \leq m, n \leq N - 2$. Also, let $\boldsymbol{a}(f) := \boldsymbol{a}(N, f) := [1, e^{i2\pi f}, \ldots, e^{i2\pi(N-1)f}]^T$ denote a size-N discrete complex sinusoid with frequency $f \in \mathbb{T}$.