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Slepian-Bangs-type formulas and the related Misspecified Cramér-Rao Bounds for Complex Elliptically Symmetric distributions

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ABSTRACT

In this paper, Slepian-Bangs-type formulas for Complex Elliptically Symmetric distributed (CES) data vectors in the presence of model misspecification are provided. The basic Slepian-Bangs (SB) formula has been introduced in the array processing literature as a convenient and compact representation of the Fisher Information Matrix (FIM) for parameter estimation under (parametric) Gaussian data model. Extending recent results on this topic, in this paper, we provide a new generalization of the classical SB formula to parametric estimation problems involving non-Gaussian, heavy-tailed, CES distributed data in the presence of model misspecification. Moreover, we show that our proposed formulas encompass the special cases of the SB formula for CES distributions under perfect model specification, the SB formulas in the presence of misspecified Gaussian models, and the SB formula for the estimation of the scatter matrix of a set of CES distributed data under misspecification of the density generator.

1. Introduction

The asymptotic performance analysis of an estimation algorithm mostly relies on two simplified assumptions: *i*) the data are assumed to be Gaussian distributed and *ii*) the data model used to derive the estimation algorithm is supposed to be *correctly specified*, that is the probability density function (pdf) assumed to derive an estimator of the parameters of interest and the true pdf that statistically characterizes the data are exactly the same.

Although these assumptions guarantee the possibility to perform "elegant" performance assessment, e.g. by evaluating the Cramér-Rao Bound (CRB) for the estimation problem at hand and/or by obtaining a closed form expression for the Mean Square Error (MSE) of a given estimator, the everyday engineering practice clearly calls the hypotheses *i*) and *ii*) into question. Regarding the Gaussian model assumption, large-scale measurement campaigns and the subsequent statistical analysis of the data gathered from a plethora of engineering applications, e.g. outdoor/indoor mobile communications, radar/sonar systems or magnetic resonance imaging (MRI), have highlighted the impulsive, heavy-tailed behaviour of the observations [1]. These experimental evidences have motivated the need to go beyond the Gaussian model and develop new statistical models able to better characterize the data. One of the more flexible and general non-Gaussian model is represented by the set of the Complex Elliptically Symmetric (CES) distributions [2], also called Multivariate Elliptically Contoured distributions [3]. CES distributions encompasses the complex Gaussian, the Generalized Gaussian and all the Compound Gaussian (CG) distributions, such as the complex *t*-distribution and the *K*-distribution, as special cases. The pdf of a CES distributed N-dimensional random vector $\mathbf{x}_l \in \mathbb{C}^N$ is completely characterized by the mean value $\boldsymbol{\gamma}$, the scatter (or shape) matrix $\boldsymbol{\Pi}$ and by a real valued function $w(t): \mathbb{R}^+ \to \mathbb{R}$, called the *density generator*, i.e. $\mathbf{x}_l \sim CES_N(\boldsymbol{\gamma}, \boldsymbol{\Pi}, w)$ [2,3]. The CES distributions have been used in a variety of applications, in particular in the radar and array signal processing fields.

Other experimental evidences reveal recurring violations of the matched model assumption, that is the claim of a perfect match between the assumed and the true data model. The mathematical bases of a formal theory of the parameter estimation under model misspecification has been firstly developed by statisticians as Huber [4], White [5] and Vuong [6] and recently rediscovered by the Signal Processing (SP) community [7–9] and applied to a va-

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riety of well-known engineering problems: to Direction-of-Arrival (DoA) estimation in array and MIMO processing [7,10], to covariance/scatter matrix estimation in CES distributed data [8,11,12], to radar-communication systems coexistence [13] and to waveform parameter estimation in the presence of uncertainty in the propagation model [14], just to name a few.

This brief discussion clearly highlights the need to overtake both the Gaussian and the matched model assumptions while assessing the (asymptotic) performance of an estimator. As extensively discussed in the SP literature, one of the main tool for the performance assessment is the CRB that provides a lower bound to the MSE achievable by any unbiased estimator for a given estimation problem (see e.g. [15]). Under the matched model assumption, the CRB can be evaluated as the inverse of the Fisher Information Matrix (FIM), then having a convenient and easy way to evaluate the FIM would be of great practical utility. To this end, in array processing applications, the celebrated Slepian-Bangs (SB) formula has been introduced. Developed in the seminal works of Slepian [16] and Bangs [17], the SB formula provides a useful and compact expression of the FIM for vector parameter estimation under both Gaussian and matched model assumptions [15, Chapter 3, Appendix 3C]. Specifically, let $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d$ be a d-dimensional, deterministic parameter vector and let $\mathbf{x} = \{\mathbf{x}_l\}_{l=1}^L$ with $\mathbf{x}_l \in \mathbb{C}^N$, be a set of L *independent* (possibly) complex random vectors, usually called *snapshots*, representing the available observations. If we assume that each snapshot follows a (complex) Gaussian parametric model, such that $\mathbf{x}_l \sim \mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta}))$, then the FIM for the estimation of $\boldsymbol{\theta} \in \Theta$ can be expressed by means of the SB formula.

The first generalization of the SB formula to a non-Gaussian, but still perfectly matched, data model has been proposed by Besson and Abramovich in [18]. Specifically, Besson and Abramovich derived a compact expression for the FIM for the estimation of $\boldsymbol{\theta} \in \Theta$ when each snapshot $\mathbf{x}_l$ is characterized by a parametric CES distribution, i.e. $\mathbf{x}_l \sim CES_N(\boldsymbol{\gamma}_l(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta}), w)$. Note that the functional form of the parametrized mean value $\boldsymbol{\gamma}_l(\boldsymbol{\theta})$ is allowed to vary from snapshot to snapshot, while the covariance matrix is assumed to be constant. Clearly, since the Gaussian model belongs to the CES class, this generalized SB formula collapses to the classical one if the data are Gaussian distributed.

The second important step ahead has been made by Richmond and Horowitz in [7] and then by Parker and Richmond in [14] where the classical, Gaussian-based, SB formula has been extended to estimation problems under model misspecification, i.e. when the assumed parametric Gaussian model, say $\mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta}))$, could differ from the true (possibly non parametric) one, indicated as $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. In other words, we allow the assumed parametric mean value $\boldsymbol{\gamma}(\boldsymbol{\theta})$ and the assumed parametric covariance matrix $\boldsymbol{\Pi}(\boldsymbol{\theta})$ to differ from the true $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for every possible value of the parameter vector $\boldsymbol{\theta} \in \Theta$, i.e. $\mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta})) \neq \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \forall \boldsymbol{\theta} \in \Theta$. It is worth to underline that in the estimation framework under model misspecification, the FIM loses its classical statistical sense and it has to be substituted by the matrices $\mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{B}(\boldsymbol{\theta})$ defined in [8], Eqs. (1) and (7), respectively (see also [6,7]). Consequently, in this context, SB-type formulas could be exploited to obtain $\mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{B}(\boldsymbol{\theta})$ needed to evaluate the counterpart of the CRB in the presence of model misspecification, i.e. the Misspecified CRB (MCRB) [4,6–8,11]. In particular, in [7] the authors derived SB-type formulas for the "decoupled" scenario in which the unknown parameter vector $\boldsymbol{\theta} \in \Theta$ can be partitioned in two sub-vectors $\boldsymbol{\eta}$ and $\boldsymbol{\nu}$, named "deterministic" and "stochastic" parameter sub-vectors respectively, such that $\boldsymbol{\theta} = [\boldsymbol{\eta}^T, \boldsymbol{\nu}^T]^T$ and $\mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta})) \triangleq \mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\Pi}(\boldsymbol{\nu})) \neq \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \forall \boldsymbol{\theta} \in \Theta$. The findings presented in [7] have been extended in [14] to include the coupling of deterministic and stochastic parameters. More formally, in [14], SB-type formulas have been derived for the following misspecified scenario $\mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\theta}), \boldsymbol{\Pi}(\boldsymbol{\theta})) \triangleq \mathcal{CN}(\boldsymbol{\gamma}(\boldsymbol{\eta}, \boldsymbol{\omega}), \boldsymbol{\Pi}(\boldsymbol{\nu}, \boldsymbol{\omega})) \neq \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \forall \boldsymbol{\theta} \in \Theta$ where the unknown parameter vector $\boldsymbol{\theta} \in \Theta$ is partitioned as $\boldsymbol{\theta} = [\boldsymbol{\eta}^T, \boldsymbol{\nu}^T, \boldsymbol{\omega}^T]^T$.

The natural extension of the works of Besson and Abramovich [18], Richmond and Horowitz [7] and Parker and Richmond [14] would be to derive SB-type formulas for parametric estimation problems involving CES distributed data under model misspecification. This paper aims exactly at filling this gap and obtaining some general "misspecified" SB formulas for CES distributed data.

Remark: Throughout this paper, we consider only the case of *real* parameter vectors. This is not a limitation, since we can always maps a complex vector in a real one simply by stacking its real and the imaginary parts. Clearly, the proposed derivation of the SB-type formulas could also be developed directly in the complex field by means of the Wirtinger calculus as in [7,19].

Notation: Throughout this paper, italics indicates scalar quantities (a, A), lower case and upper case boldface indicate column vectors ($\mathbf{a}$) and matrices ($\mathbf{A}$) respectively. Each entry of a matrix $\mathbf{A}$ is indicated as $a_{i,j} \triangleq [\mathbf{A}]_{i,j}$. * indicates the complex conjugation. The superscripts T and H indicates the transpose and the Hermitian operators, then $\mathbf{A}^H = (\mathbf{A}^*)^T$. Let $f(t)$ be a real scalar function, than $f'(t) \triangleq df(t)/dt$. Let $\mathbf{A}(\boldsymbol{\theta})$ be a matrix (or possibly vector or even scalar) function of the vector $\boldsymbol{\theta}$, then $\mathbf{A}_0 \triangleq \mathbf{A}(\boldsymbol{\theta}_0)$ while $\mathbf{A}_i^0 \triangleq \frac{\partial \mathbf{A}(\boldsymbol{\theta})}{\partial \theta_i}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$ and $\mathbf{A}_{ij}^0 \triangleq \frac{\partial^2 \mathbf{A}(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$, where the vector $\boldsymbol{\theta}_0$ will be always explicitly defined in the paper. For two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semi-definite. Finally, for random variables or vectors, the notation $=_d$ stands for "has the same distribution as".

2. Problem setup

Let $\mathbf{x} = \{\mathbf{x}_l\}_{l=1}^L$, with $\mathbf{x}_l \in \mathbb{C}^N$, be a set of L independent complex random vectors (or *snapshots*) representing the available observations. We assume that each snapshot is sampled from a CES distribution [2,3], i.e., $\mathbf{x}_l \sim CES_N(\boldsymbol{\mu}_l, \boldsymbol{\Sigma}, g)$, then its pdf can be expressed as:

$$p_X(\mathbf{x}_l) \triangleq p_X(\mathbf{x}_l; \boldsymbol{\mu}_l, \boldsymbol{\Sigma}) = c_{N,g} |\boldsymbol{\Sigma}|^{-1} g((\mathbf{x}_l - \boldsymbol{\mu}_l)^H \boldsymbol{\Sigma}^{-1} (\mathbf{x}_l - \boldsymbol{\mu}_l)) \quad (1)$$

where $c_{N,g}$ is a normalizing constant, $g(t): \mathbb{R}^+ \to \mathbb{R}$ is the *density generator*, $\boldsymbol{\mu}_l = E_p\{\mathbf{x}_l\}$ is the mean value and $\boldsymbol{\Sigma}$ is a positive definite Hermitian matrix called *scatter matrix*. In the rest of this paper, we always assume that the scatter matrix $\boldsymbol{\Sigma}$ is of full rank, i.e. $\text{rank}(\boldsymbol{\Sigma}) = N$. From the Stochastic Representation Theorem [2], a CES distributed random vector can be expressed as:

$$\mathbf{x}_l =_d \boldsymbol{\mu}_l + \mathcal{R}\mathbf{T}\mathbf{u}_l, \quad (2)$$

where:

- $\mathbf{u}_l \sim U(\mathbb{C}S^N)$ is a N-dimensional complex random vector uniformly distributed on the unit hyper-sphere with $N-1$ topological dimension. As reported in [2] (Lemma 1), $E_p\{\mathbf{u}_l\} = 0$ and $E_p\left\{\mathbf{u}_l\mathbf{u}_l^H\right\} = (1/N)\mathbf{I}$ where $\mathbf{I}$ is the identity matrix of a suitable dimension.
- $\mathcal{R} \triangleq \sqrt{\mathcal{Q}}$ is a real and non-negative random variable called *modular variate*, while $\mathcal{Q}$ is called *second order modular variate*. Moreover, under the assumption that $\text{rank}(\boldsymbol{\Sigma}) = N$, we have that:

$$Q_l \triangleq (\mathbf{x}_l - \boldsymbol{\mu}_l)^H \boldsymbol{\Sigma}^{-1} (\mathbf{x}_l - \boldsymbol{\mu}_l) =_d \mathcal{Q}, \forall l \in \mathbb{N}. \quad (3)$$

As shown in [2], the pdf of $\mathcal{Q}$ has a one-to-one relation with density generator:

$$p_{\mathcal{Q}}(t) = \delta_{N,g}^{-1} t^{N-1} g(t), \quad (4)$$

where $\delta_{N,g} \triangleq \int_0^\infty t^{N-1} g(t) dt < \infty$. As a consequence of (3) and (4), the expectation of functions of the quadratic form Q_l, say