Available online at www.sciencedirect.com

ScienceDirect

Comput. Methods Appl. Mech. Engrg. 289 (2015) 104–130

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

Analysis of an augmented pseudostress-based mixed formulation for a nonlinear Brinkman model of porous media flow[☆]

Gabriel N. Gatica[a,*], Luis F. Gatica[b], Filánder A. Sequeira[c,1]

[a] CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

[b] CI^{2}MA - Universidad de Concepción, and Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile

[c] Escuela de Matemática, Universidad Nacional de Costa Rica, Heredia, Costa Rica

Received 28 October 2014; received in revised form 24 January 2015; accepted 30 January 2015
Available online 11 February 2015

Highlights

- A new augmented mixed finite element method for the 2D nonlinear Brinkman model.
- Dual-mixed formulation with gradient of velocity and pseudostress as main unknowns.
- Velocity and pressure are easily recovered through a simple postprocessing.
- Neumann boundary conditions are imposed weakly.
- A reliable and efficient residual-based a posteriori error estimator is provided.

Abstract

In this paper we introduce and analyze an augmented mixed finite element method for the two-dimensional nonlinear Brinkman model of porous media flow with mixed boundary conditions. More precisely, we extend a previous approach for the respective linear model to the present nonlinear case, and employ a dual-mixed formulation in which the main unknowns are given by the gradient of the velocity and the pseudostress. In this way, and similarly as before, the original velocity and pressure unknowns are easily recovered through a simple postprocessing. In addition, since the Neumann boundary condition becomes essential, we impose it in a weak sense, which yields the introduction of the trace of the fluid velocity over the Neumann boundary as the associated Lagrange multiplier. We apply known results from nonlinear functional analysis to prove that the corresponding continuous and discrete schemes are well-posed. In particular, a feasible choice of finite element subspaces is given by Raviart–Thomas elements of order $k \geq 0$ for the pseudostress, piecewise polynomials of degree $\leq k$ for the gradient of the velocity, and continuous piecewise polynomials of degree $\leq k+1$ for the Lagrange multiplier. We also derive a reliable and

☆ This work was partially supported by CONICYT-Chile through BASAL project CMM, Universidad de Chile, project Anillo ACT1118 (ANANUM), and the Becas-Chile Programme for foreign students; by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción; and by Dirección de Investigación of the Universidad Católica de la Santísima Concepción, through the project DIN 07/2014.

* Corresponding author.

E-mail addresses: ggatica@ci2ma.udec.cl (G.N. Gatica), lgatica@ucsc.cl (L.F. Gatica), filander.sequeira@una.cr, fsequeira@ci2ma.udec.cl (F.A. Sequeira).

[1] Present address: CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

http://dx.doi.org/10.1016/j.cma.2015.01.019

efficient residual-based a posteriori error estimator for this problem. Finally, several numerical results illustrating the performance and the robustness of the method, confirming the theoretical properties of the estimator, and showing the behavior of the associated adaptive algorithm, are provided.

Keywords: Nonlinear Brinkman model; Mixed finite element method; Augmented formulation; High-order approximations

1. Introduction

The Brinkman model of porous media flow, which can be seen as a mixture of Darcy's and Stokes' equations, is usually hard to solve, firstly because of the wide range of possible permeability ratios, and secondly due to the nature of the mixed boundary conditions involved. One way of solving the first issue is by means of stabilized methods (see, e.g. [1], [2]), whereas the weak imposition of the Dirichlet boundary conditions, using Nitsche's method, has been applied recently to deal with the second difficulty (see, e.g. [3] and the references therein). However, most of the variational formulations found in the literature are based on the typical Stokes-type (also called primal-mixed) approach in which the velocity and the pressure are kept as the main unknowns. Actually, up to the authors' knowledge, no stress-based or pseudostress-based approaches seemed to be available until the recent contribution [4], where an alternative way of dealing with the mixed boundary conditions and the a priori and a posteriori error analyses of a dual-mixed approach for the two-dimensional Brinkman problem were provided. Indeed, the pseudostress $\boldsymbol{\sigma}$ is the main unknown of the resulting saddle point problem in [4], and the velocity and pressure are easily recovered in terms of $\boldsymbol{\sigma}$ through simple postprocessing formulae. In addition, as it is usual for dual-mixed methods, the Dirichlet boundary condition for the velocity becomes natural in this case, and the Neumann boundary condition, being essential, is imposed weakly through the introduction of the trace of the velocity on that boundary as the associated Lagrange multiplier. In this way, the Babuška–Brezzi theory is applied first in [4] to establish sufficient conditions for the well-posedness of the resulting continuous and discrete formulations. In particular, a feasible choice of finite element subspaces is given by Raviart–Thomas elements of order $k \geq 0$ for the pseudostress, and continuous piecewise polynomials of degree $k+1$ for the Lagrange multiplier. Next, a reliable and efficient residual-based a posteriori error estimator is derived there. Suitable auxiliary problems, the continuous inf–sup conditions satisfied by the bilinear forms involved, a discrete Helmholtz decomposition, and the local approximation properties of the Raviart–Thomas and Clément interpolation operators are the main tools for proving the reliability. In turn, Helmholtz's decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions are employed to show the efficiency.

The purpose of the present paper is to extend the analysis and results from [4] to a class of Brinkman models whose viscosity depends nonlinearly on the gradient of the velocity, which is a characteristic feature of quasi-Newtonian Stokes flows (see, e.g. [5–7]). To this end, we introduce the gradient of the velocity as a new unknown and follow the approach from [6] to deal with the aforedescribed nonlinearity. Moreover, in order to be able to apply the abstract theory from [8] dealing with nonlinear saddle point problems (see also [9,10]), we need to modify the resulting variational formulation by augmenting it with a redundant equation arising from the constitutive law relating the pseudostress and the velocity gradient. The rest of this work is organized as follows. In Section 2 we define our nonlinear Brinkman model. Then, in Section 3 we introduce the augmented continuous formulation and analyze its solvability. The associated mixed finite element method is introduced and analyzed in Section 4. Next, in Section 5 we basically apply the techniques from [11,12], and [6], to derive a reliable and efficient residual-based a posteriori error estimator for our Galerkin scheme. Finally, some numerical results showing the good performance and robustness of the mixed finite element method, confirming the reliability and efficiency of the estimator, and illustrating the behavior of the associated adaptive algorithm are reported in Section 6.

We end this section with some notations to be used below. Given $\boldsymbol{\tau} := (\tau_{ij}), \boldsymbol{\zeta} := (\zeta_{ij}) \in R^{2\times 2}$, we write as usual

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji}), \qquad \mathrm{tr}\,(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}, \qquad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij},$$