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Nonconvex and nonsmooth total generalized variation model for image restoration

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ABSTRACT

In this paper, we propose a nonconvex and nonsmooth total generalized variation (TGV) model for image restoration, which can provide an even sparser representation of the variation of the image function than the traditional TGV model that uses convex l_1 norm to measure the variation. New model combines the advantages of nonconvex regularization and TGV regularization, and can preserve image edges well and simultaneously alleviate the staircase effects often arising in the total variation based models. Two different iteratively reweighed algorithms are introduced to numerically solve the proposed nonconvex and nonsmooth TGV model. Numerical results show that the proposed model is effective in edge-preserving and staircase-reduction in image restoration. In addition, compared with several state-of-the-art variational models, the proposed model has the best performance in terms of PSNR and MSSIM values.

1. Introduction

Image restoration and reconstruction is an important task in image processing and computer version, which has been applied in various areas such as medical imaging, pattern recognition, video coding and so on. So far, lots of techniques have been developed for image restoration and reconstruction such as spatial filtering [1,2], transform domain filtering [3,4], partial differential equation (PDE) modeling [5,6], and variational methods [7]. It should be pointed out that with the development of computational software and hardware technology, machine learning based methods [8–11] have received extensive attention in recent years, such as deep learning [8,11], linear regression [9], Bayesian learning [10,11], and son on.

In this paper, we focus on the image restoration from a noisy version by using variational method, where the noisy image is obtained by adding the white noise with zero-mean to the clean data, modeled as

$$f = u + n \tag{1.1}$$

where $u : \Omega \to \mathbb{R}$ represents the true image, n is additive white noise, and f is the corresponding noisy version. It is well known that solving the true image u from the linear system (1.1) is ill-posed [12,13]. To tackle this problem, one of the common methods is regularization technique that minimizes cost functional to obtain the stable and accurate solutions [14–16]. Specifically, regularization technique implements image restoration by solving the following variational problem,

$$\min_{u\in U} \{E(u) = \lambda D(u, f) + R(u)\} \tag{1.2}$$

where the first term $D(u, f)$ is fidelity term that penalizes the restoration u to be very far away form the original observation f, the second term $R(u)$ is a regularization term which represents the prior information about the object to be restored, such as continuity, smoothness or bounded variation, and λ is a tuning parameter that controls the balance between the fidelity term and the regularization term.

How to choose an effective regularization term $R(u)$ is the key problem in variational image restoration. The earliest regularization term is a quadric functional of the L^2 norm, $\|Lu\|_2^2$, proposed by Phillips [17] and Tikhonov [18] in 1960s, often called Tikhonov regularization, where L is identity operator or differential operator. Tikhonov regularization has excellent performance in noise-removing. However, it often overly smoothes the image edges which are important features in image recognition [19,20]. Later on, total variation (TV) regularization was proposed by Rudin et al. [19] in 1992 to conquer this problem. TV regularization allows the solution with discontinuities along curves, so edges and

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contours can be preserved in the restoration. However, it often yields the undesired staircase effects in the smooth regions of the restoration, since it tends to transform the smooth regions of the solution into piecewise constant regions during the functional minimizing [21,22]. To overcome this drawback, many improved TV based regularization methods are proposed, such as high-order TV [23–26], hybrid TV [27], non-local TV [28,29], overlapping TV [30,31], anisotropic TV [32–34], fractional order TV [35], and so on. In this paper, we focus on the total generalized variation (TGV) [23,26] regularization that is a generalization of TV. TGV regularization can reconstruct image features up to an arbitrary order differentiation, such as piecewise constant, piecewise affine, piecewise quadratic and so on, so it has the superior performance in image reconstruction than TV based regularization models which only can reconstruct piecewise constant regions. In other word, TGV regularization model can reduce the staircase effect that often arises in TV based models. The above regularization terms all are convex, which ensures the existence and uniqueness of the solution. In addition, they can be efficiently solved by many convex optimization algorithms.

In the last decades, signal sparsity-prior has received scholars' constant attention, which simulates the human visual system. It is based on the observation that signals (also images) usually have a sparse representation in some transformed domain (such as Fourier transform, cosine transform), or some dictionaries (such as wavelet dictionary, framelet dictionary, self-adaptive dictionary) [36,37]. It is well known that nonconvex norms are more suitable to measure the sparsity than the corresponding convex ones, since they are much closer to the l_0 norm that is exactly the measure of sparsity [38–40]. Since the seminal work of Geman and Geman in [41], various nonconvex regularization models have been proposed, such as [42–44]. Although the existence and the uniqueness of the solution for nonconvex regularization models still are open questions, a variety of applications (e.g., [44–47]) have shown that nonconvex regularization models can recover the images of high quality with sharp and neat edges. The authors in [42,47] provided a theoretical explanation for this phenomenon. The nonconvex regularization has the advantage in edge-preserving, but it leads to very serious staircase effects, even more severe than TV based models.

In this paper, combining the advantages of TGV regularization and nonconvex regularization, and avoiding their main drawbacks, we propose a nonconvex and nonsmooth TGV (NNTGV) model for image restoration. New model can preserve image edges well and simultaneously alleviate the staircase effects. In addition, based on the majorization-minimization (MM) scheme [48], two iteratively reweighted algorithms are introduced to numerically solve the proposed model: iteratively reweighted least squares (IRLS) algorithm [49–51] and iteratively reweighted l_1 (IRL1) algorithm [52,53], respectively. The main contributions of this paper are summarized as follows: (1) A nonconvex and nonsmooth TGV model is proposed for image restoration. (2) Two iteratively reweighted algorithms are introduced to solve the proposed nonconvex minimization problem. (3) We conduct extensive experiments to verify the proposed model in comparison with several state-of-the-art variational models.

The rest of the paper is organized as follows. In Section 2, we give a brief review of TGV, nonconvex and nonsmooth TV, and iteratively reweighed algorithm, which are very relevant to our present study. In Section 3, we present the proposed NNTGV model, and derive two efficient iteratively reweighted algorithms to solve it. In Section 4, we show some numerical experiments to demonstrate the effectiveness of the proposed model. In addition, we compare it with several state-of-the-art variational models to show the superior performance in staircase-reduction and edge-preserving. The paper is summarized in Section 5.

2. Background knowledge

2.1. TGV model

TGV is a generalization of TV, which is defined on the dual formulation incorporating the space of symmetric k-tensors that are defined as

$$\mathrm{Sym}^k(\mathbb{R}^d) = \left\{\xi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R} : \xi \text{ is } k\text{-linear and symmetric}\right\}$$

Let $\Omega \subset \mathbb{R}^2$ be image domain and $u : \Omega \to \mathbb{R}$ be a image function. Then, the TGV of order $k \geq 1$ with positive weights $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ is defined as

$$\mathrm{TGV}_{\boldsymbol{\alpha}}^k(u) = \sup\left\{\int_\Omega u \mathrm{div}^k \boldsymbol{v} dx \;\middle|\; \boldsymbol{v} \in C_c^k(\mathrm{Sym}^k(\Omega, \mathbb{R}^2)), \; ||\mathrm{div}^l \boldsymbol{v}||_\infty \leqslant \alpha_l, \; l = 0, \ldots, k-1\right\}$$

where $C_c^k(\mathrm{Sym}^k(\Omega, \mathbb{R}^2))$ is the space of compactly supported symmetric k-tensor fields, $\|\cdot\|_\infty$ is the L^∞ norm, and div^k is the generalization of the divergence operator of k order to the tensor field. It is obviously that TGV is a generalization of TV. When $k=1$, $\boldsymbol{\alpha} = (1)$ and $\mathrm{Sym}^1(\mathbb{R}^2) = \mathbb{R}^2$, then the definition of TGV is actually the dual formulation of TV, i.e., $\mathrm{TGV}_1^1(u) = \mathrm{TV}(u)$. We note that the $\mathrm{TGV}_{\boldsymbol{\alpha}}^k$ of all polynomials with degree less than or equivalent to $k-1$ is zero. So minimizing $\mathrm{TGV}_{\boldsymbol{\alpha}}^k(u)$ will lead to the piecewise polynomial solutions. The second order TGV is often applied in variational image restoration, which is defined as

$$\mathrm{TGV}_{\boldsymbol{\alpha}}^2(u) = \sup\left\{\int_\Omega u \mathrm{div}^2 \boldsymbol{v} dx \;\middle|\; \boldsymbol{v} \in C_c^2(\mathrm{Sym}^2(\Omega, \mathbb{R}^2)), \; ||\boldsymbol{v}||_\infty \leqslant \alpha_0, ||\mathrm{div}\boldsymbol{v}||_\infty \leqslant \alpha_1\right\} \tag{2.1}$$

Minimizing $\mathrm{TGV}_{\boldsymbol{\alpha}}^2(u)$ with respect to u can reconstruct the piecewise constant and piecewise linear functions, so staircase effects can be alleviated. But, the $\mathrm{TGV}_{\boldsymbol{\alpha}}^2(u)$ formulated as in (2.1) is difficult to solved in the practice. In this study, we use another formulation of the second order TGV in terms of l_1 minimization. Firstly, for notational convenience, we define three function spaces U, W, and S as

$$U \overset{\text{def}}{=} C_c^2(\Omega, \mathbb{R}), \quad W \overset{\text{def}}{=} C_c^2\left(\Omega, \mathbb{R}^2\right), \quad S \overset{\text{def}}{=} C_c^2\left(\Omega, \mathbb{R}^4\right)$$

Let $u \in U$, $\boldsymbol{w} = (w^1, w^2)^T \in W$, and $\boldsymbol{s} = (s^1, s^2, s^3, s^4)^T \in S$, in which $w^i \in U (i = 1, 2)$, and $s^j \in U (j = 1, \ldots, 4)$. Here the 2-tensor $\boldsymbol{s}$ is converted into a vector column by column for computational convenience. The gradient operators on the space U and W, and the divergence operators on the space W and S are defined as

$$\nabla_u : U \to W, \quad \nabla_u u = (\partial_x^+ u, \partial_y^+ u)^T$$

$$\nabla_w : W \to S, \quad \nabla_w \boldsymbol{w} = (\partial_x^+ w^1, \frac{1}{2}(\partial_y^+ w^1 + \partial_x^+ w^2), \frac{1}{2}(\partial_y^+ w^1 + \partial_x^+ w^2), \partial_y^+ w^2)^T$$

$$\mathrm{div}_w : W \to U, \quad \mathrm{div}_w \boldsymbol{w} = \partial_x^- w^1 + \partial_y^- w^2$$

$$\mathrm{div}_s : S \to W, \quad \mathrm{div}_s \boldsymbol{s} = \left(\partial_x^- s^1 + \partial_y^- s^2, \partial_x^- s^3 + \partial_y^- s^4\right)^T$$

where ∂_x^+, ∂_y^+, ∂_x^- and ∂_y^- are the first order forward and backward discrete derivation operators in the x-direction and y-direction, respectively, which are defined as

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j}, & \text{if } j < N \\ 0, & \text{if } j = N \end{cases}$$