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# Rapid accurate frequency estimation of multiple resolved exponentials in noise

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ABSTRACT

The estimation of the frequencies of the sum of multiple resolved exponentials in noise is an important problem due to its application in diverse areas from engineering to chemistry. Yet to date, no low cost Fourier-based algorithm has been successful at obtaining unbiased estimates that achieve the Cramér–Rao lower bound (CRLB) over a wide range of signal-to-noise ratios. In this work, we achieve precisely this goal, proposing a fast yet accurate estimator that combines an iterative frequency-domain interpolation step with a leakage subtraction scheme. By analysing the asymptotic performance and the convergence behaviour of the estimator, we show that the estimate of each frequency converges to the asymptotic fixed point. Thus, the estimator is asymptotically unbiased and the variance is extremely close to the CRLB. We verify the theoretical analysis by extensive simulations, and demonstrate that the proposed algorithm is capable of obtaining more accurate estimates than state-of-the-art high resolution methods while requiring significantly less computational effort.



## 1. Introduction

Estimating the frequencies of the components in sums of complex exponentials in noise is an important research problem as it arises in many applications such as radar, wireless communications and spectroscopy analysis [1–5]. The signal model given by

$$x(n) = \sum_{l=1}^{L} s_l(n) + w(n) = \sum_{l=1}^{L} A_l e^{j2\pi f_l n} + w(n), \quad n = 0 \ldots N-1. \tag{1}$$

Here $N$ is the number of time samples. $L$ is the number of components. $f_l \in [-0.5, 0.5]$ is the normalised frequency of the $l$th component. The noise terms $w(n)$ are additive Gaussian noise with zero mean and variance $\sigma^2$. The signal to noise ratio (SNR) of the $l$th component is $\rho_l = |A_l|^2/\sigma^2$.

In this work, we assume that the components are resolved and the goal is to estimate their frequencies cheaply yet accurately. The estimation is achieved by assuming $L$ to be known *a priori*. The estimation of $L$ is beyond the scope of this paper as it is widely considered as a separate problem [6–9]. Nevertheless, in the case where estimating $L$ is necessary, model order determination methods such as the Akaike information criterion (AIC) [10] and the minimum description length (MDL) [11] can be used. Particularly, when $L \ll N$, the degrees of freedom employed in these methods can be much less than $\lfloor N/3 \rfloor$ resulting in a significantly reduced computational cost of the model order determination step [12].

The estimation of the frequencies of multi-tone exponentials has been the subject of intense research for many decades. The various algorithms that have been proposed to handle it, [13,14], can be categorised into two types: non-parametric estimators and parametric estimators.

Non-parametric spectral estimators, including the traditional Capon [15], APES [16] and IAA [17] can be used to estimate the component frequencies using peak picking on the spectral estimates without knowing the number of components. These algorithms exhibit a high resolution, meaning that they can resolve closely separated components, by consuming $O(N^2 + K\log_2 K)$ for the computation of a length-$K$ spectrum [18–20]. However, they obtain accurate estimates at the expense of very high computational cost as the peak picking needs to be performed in a very dense spectrum grid where $K \gg N$. Consequently, the high computational burden makes them poor approaches for resolved components where the computational cost to achieve the CRLB can become prohibitive.

Instead of estimating the signal spectrum, parametric estimators try to find accurate estimates of the signal parameters only. They can be further classified into time and frequency domain approaches. The time domain approaches include subspace methods such as matrix pencil (MP) [21] and ESPRIT [22,23], which use the singular value decomposition (SVD) to separate the noisy signal into pure signal and noise subspaces, or iterative optimisation algorithms including IQML [24] and weighted least

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squares (WLS) [25,26] that minimise the error between the noisy and pure signals subject to different constraints. However, similar to non-parametric estimators, they suffer from high computational cost due to the SVD operation, matrix inversion and/or the eigenvalue decomposition involved, which require $O(N^3)$ for computation for large $N$. In [5], the author proposed a computationally efficient ML-based algorithm that is of smaller computational complexity by exploiting the Vandermonde structure of the signal mode. Nevertheless, relatively heavy calculations are still required to achieve high accuracy when the number of components is large compared with the signal length. In frequency-domain parametric estimators, on the other hand, are generally computationally more efficient than time-domain methods. These include the traditional CLEAN approach [27] and the RELAX algorithm [7], which combine the maximiser of the discrete periodogram and an iterative estimation–subtraction procedure. But their estimation error is of the same order as the reciprocal of the size of the discrete periodogram [28], which make them biased when a sparse spectrum is calculated, or computationally complex for obtaining a dense spectrum. A number of algorithms have been proposed in [29–32] to refine the maximiser of the $N$-point periodogram. But, as these are developed for single-tone signals, they perform poorly in the multiple component case due to the bias resulting from the interference of the components with one another. Much work, [1,33,6,3,4], has been done to reduce the effect of the interference by either applying the interpolators after pre-multiplying the signal by a time domain window. However, non-rectangular windows lead to deterioration of estimation accuracy.

To summarise, there has not been to date any successful algorithm that is capable of achieving unbiased estimates close to the CRLB while at the same time maintaining high computational efficiency. In this paper, we solve this problem by proposing a novel parametric estimation algorithm that operates in the frequency domain and achieves excellent performance. The new approach is more computationally efficient than the non-parametric and time domain parametric estimators, yet it outperforms them in terms of estimation accuracy.

The rest of the paper is organised as follows. In Section 2, we present the novel frequency estimation algorithm. We analyse the algorithm in Section 3 and give its theoretical performance. In Section 4, we show simulation results by comparing the proposed algorithm with state-of-art parametric estimators and the Cramér–Rao lower bound (CRLB). Finally, some conclusions are drawn in Section 5.

## 2. The proposed method

Let $\hat{\lambda}$ denote the estimate of the parameter $\lambda$. The A&M estimator of [30,28] is a powerful and efficient algorithm for the estimation of the frequency of a single-tone signal. It uses a two stage strategy, obtaining first a coarse estimate from the maximum of the $N$-point periodogram

$$\hat{m} = \underset{k}{\operatorname{argmax}}\,|X(k)|^2 = \underset{k}{\operatorname{argmax}}\left|\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}\right|^2. \tag{2}$$

In the noiseless case, we have $\lim_{N\to\infty}\hat{m} = m$, a.s. [30]. When $\hat{m}$ is the true maximum bin, the frequency of the signal can be written as

$$f = \frac{\hat{m} + \delta}{N}, \tag{3}$$

where $\delta \in [-0.5, 0.5]$ is the frequency residual. The A&M algorithm then refines the coarse estimate by interpolating on Fourier coefficients to obtain an estimate for the residual $\delta$.

Let $X_{\pm 0.5}$ be the Fourier coefficients at locations $\hat{m} \pm 0.5$. In the noiseless case, these are given by

$$X_{\pm 0.5} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}\left(\hat{m}\pm 0.5\right)n} = \frac{A}{N}\frac{1 + e^{j2\pi\delta}}{1 - e^{j\frac{2\pi}{N}\left(\delta\mp 0.5\right)}}. \tag{4}$$

Putting $z^{-1} = e^{-j2\pi\frac{\delta}{N}}$, an estimate of which is constructed as

$$\hat{z}^{-1} = \cos\left(\frac{\pi}{N}\right) - 2jh\sin\left(\frac{\pi}{N}\right), \tag{5}$$

where $h$ is the interpolation function

$$h = \frac{1}{2}\frac{X_{0.5} + X_{-0.5}}{X_{0.5} - X_{-0.5}}. \tag{6}$$

From $\hat{z}^{-1}$, estimates of $\delta$ and consequently of the frequency $f$ become

$$\hat{\delta} = -\frac{N}{2\pi}\Im\left\{\ln\hat{z}^{-1}\right\}, \quad \text{and} \quad \hat{f} = \frac{\hat{m} + \hat{\delta}}{N}. \tag{7}$$

Here $\Im\{\bullet\}$ denotes the imaginary part of $\bullet$. The key to the A&M algorithm compared to other interpolators like those of [34] is that it can be implemented iteratively in order to improve the estimation accuracy [30]. In each iteration the estimator removes the previous estimate of the residual before re-calculating Fourier coefficients and re-interpolating. It was shown in [30] that two iterations are sufficient for the estimator to obtain asymptotically unbiased frequency estimate with the variance only 1.0147 times the CRLB.

Now turning to the multi-tone case, that is $L \geq 2$, let $\{\hat{m}_l\}_{l=1}^{L}$ be the estimates of the maximum bins. Also let $\hat{\delta}_p$ be the estimates of the residuals from the previous iteration. The Fourier coefficients of the $p$th component at locations $(\hat{m}_p + \hat{\delta}_p \pm 0.5)$ are

$$X_{p,\pm 0.5} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}\left(\hat{m}_p+\hat{\delta}_p\pm 0.5\right)n} \tag{8}$$

$$X_{p,\pm 0.5} = S_{p,\pm 0.5} + \sum_{l=1,l\neq p}^{L} S_{l,\pm 0.5} + W_{p,\pm 0.5}, \tag{9}$$

where $S_{p,\pm 0.5}$ are Fourier coefficients for a single exponential $s_p(n)$ as per Eq. (4). $W_{p,\pm 0.5}$ are the corresponding noise terms at the interpolation locations. $S_{l,\pm 0.5}$ is the leakage term introduced by the $l$th component:

$$S_{l,\pm 0.5} = \frac{1}{N}\sum_{n=0}^{N-1} s_l(n)e^{-j\frac{2\pi}{N}(\hat{m}_p+\hat{\delta}_p\pm 0.5)n} = \frac{A_l}{N}\frac{1 + e^{j2\pi(\delta_l-\hat{\delta}_p)}}{1 - e^{j\frac{2\pi}{N}\left(M_{l,p}+\delta_l-\hat{\delta}_p\mp 0.5\right)}}, \tag{10}$$

where

$$M_{l,p} = \hat{m}_l - \hat{m}_p. \tag{11}$$

As proposed in [1,35], the leakage terms can be attenuated by applying a window to the signal. Although this reduces the bias it also leads to a broadening of the main lobe and comes at the cost of an increase in the estimation variance. We, on the other hand, address this problem by estimating the leakage terms in Eq. (10) and removing them in order to obtain the expected coefficients of a single exponential. We then apply the A&M estimator to estimate the frequency. It is clear from Eq. (10) that this necessitates the parameters $\delta_l$ and $A_l$ be known or at least estimates for them should be available. In what follows, we construct a procedure to achieve this.

Let us start by assuming that we have estimates $\left\{\hat{\delta}_l, \hat{A}_l\right\}_{l=1,l\neq p}^{L}$. Then the total leakage term can be obtained as