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SUPG reduced order models for convection-dominated convection–diffusion–reaction equations

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Abstract

This paper presents a Streamline-Upwind Petrov–Galerkin (SUPG) reduced order model (ROM) based on proper orthogonal decomposition (POD). This ROM is investigated theoretically and numerically for convection-dominated convection–diffusion–reaction problems. The SUPG finite element method was used on realistic meshes for computing the snapshots, leading to some noise in the POD data. Numerical analysis is used to propose the scaling of the stabilization parameter for the SUPG-ROM. Two approaches are used: One based on the underlying finite element discretization and the other one based on the POD truncation. The resulting SUPG-ROMs and the standard Galerkin ROM (G-ROM) are studied numerically. For many settings, the results obtained with the SUPG-ROMs are more accurate. Finally, one of the choices for the stabilization parameter is recommended.

Keywords: Reduced order models (ROMs); Convection-dominated equations; Streamline-Upwind Petrov–Galerkin (SUPG) method; Proper Orthogonal Decomposition (POD); Stabilization parameter

1. Introduction

The numerical simulation of many complex problems requires repeatedly solving subproblems. For instance, in the iterative solution of optimal control problems with partial differential equations, one has to solve the same type of partial differential equation, with (slightly) changing data, over and over again. With standard discretizations for partial differential equations, like finite element methods, the solution of the subproblems is often the most time-consuming part of the simulations. Reduced order modeling aims at finding low-dimensional spaces that allow solution of partial differential equations orders of magnitude more efficiently than a finite element method, with only an acceptable loss of accuracy. This paper considers reduced order models (ROMs) for scalar convection-dominated convection–diffusion–reaction equations and studies suitable choices for the parameter of a stabilized discretization.

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Solutions of convection-dominated problems usually possess layers which cannot be resolved with the underlying mesh, particularly in higher dimensions. The Galerkin finite element method is called unstable because it cannot cope with this situation and its usage leads to numerical solutions which are globally polluted with spurious oscillations. One has to use a so-called stabilized discretization. A variety of such discretizations have been proposed over the last few decades, e.g., see [1–3] for reviews and numerical comparisons. However, the question of finding a perfect discretization, i.e., a discretization which gives solutions with steep layers and without spurious oscillations, is still open. One of the most popular stabilized finite element methods is the Streamline-Upwind Petrov–Galerkin (SUPG) method proposed in [4,5]. Solutions computed with this method usually possess steep layers but also exhibit some spurious oscillations in a vicinity of the layers. The SUPG method contains a stabilization parameter whose asymptotic value for steady-state problems is well known from finite element error analysis (e.g., [6]); it depends on the local mesh width. The situation is not completely clear for time-dependent problems. For general problems, optimal estimates can only (to the best of the authors' knowledge) be derived for parameters dependent on the length of the time step. For a simplified situation, estimates can also be proven for parameters dependent on the mesh width; see [7] for details. From the practical point of view, the latter choice seems to be more appropriate since the difficulty of not being able to resolve the layers vanishes on sufficiently fine meshes but not for sufficiently small time steps.

ROMs are already used for many complex systems. One of the most popular ROM approaches is Proper Orthogonal Decomposition (POD). POD extracts the most pertinent features of a data set and naturally leads to a Galerkin formulation of a ROM, which will be denoted as the Galerkin ROM (G-ROM). This paper exclusively considers ROMs based on POD. There are situations where G-ROMs are efficient and relatively accurate (see [8–10]). However, in other situations, a G-ROM might produce inaccurate results [11]. One of the main reasons for these inaccurate results is that the underlying G-ROM can be numerically unstable, e.g., the G-ROM solution can blow up in a nonphysical way, see [12–14] for the compressible Navier–Stokes equations and [15,16] for the incompressible Navier–Stokes equations. Various stabilized ROMs have been proposed, see [12,17,18,15,19–21,16,22]; see also [23–25] for similar work in reduced basis methods. This paper focuses on ROMs for convection-dominated convection–diffusion–reaction equations and these ROMs' potential numerical instability due to the unresolved layers.

After computing the POD modes, the question is how to design a numerical method that gives a solution that is as accurate as possible in the space spanned by the first r POD modes. In the numerical studies presented below, the solution of the continuous problem is known, so accuracy always refers to this solution. When the solution of the continuous problem is not available, one could compare the ROM results with the solution of the simulation used to compute the snapshots, even if this solution is usually not the best approximation of the continuous solution in the finite element space. Since the numerical method providing the most accurate solution in the r-dimensional space is not necessarily the method which was used for computing the snapshots, other methods can be investigated as well.

The main class of methods which will be studied in this paper is SUPG stabilized ROMs (SUPG-ROMs), see, e.g., [26] for alternative approaches. To the best of the authors' knowledge the SUPG-ROM was first used in [21] and later on in [19], in both papers for the Navier–Stokes equations. In addition to the SUPG-ROMs, the G-ROM will also be investigated.

As in the finite element SUPG method, the question of appropriate stabilization parameters for SUPG-ROMs arises. SUPG-ROM parameters depending only on the spatial resolution are preferable for the same reasons as in the finite element method. A ROM based on finite element data has two parts to its spatial resolution: The spatial resolution from the finite element space and the spatial resolution from the space of POD modes used in the ROM, which is a subspace of the finite element space. One can ask upon which spatial resolution the stabilization parameter for the SUPG-ROM should depend. This is the main question studied in this paper.

The question of appropriate stabilization parameters for the SUPG-ROM is addressed by means of a numerical analysis of this problem. To the best of the authors' knowledge, the use of numerical analysis to propose the SUPG-ROM stabilization parameter is new. In the literature so far, simply the stabilization parameter from the finite element method was used, like in [21], or an optimization problem for the determination of the parameter was solved, as in [19]. Motivations for these approaches with numerical analysis were not provided. In the authors' opinion, it is important to have some support for the choice of stabilization parameters coming from numerical analysis, since parameters determined with considerations from numerical analysis should be valid for a wide range of settings (e.g., diffusion coefficients and the convection vector). Two stabilization parameters will be proposed as a result of analytical considerations. One of them is based on the finite element resolution and the other one is based on the POD spatial resolution. The resulting ROMs will be denoted as FE-SUPG-ROM and POD-SUPG-ROM, respectively.