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Short communication

Frequency estimation of multiple sinusoids with three sub-Nyquist channels☆

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ABSTRACT

Frequency estimation of multiple sinusoids is significant in both theory and application. In some application scenarios, only sub-Nyquist samples are available to estimate the frequencies. A conventional approach is to sample the signals at several lower rates. In this paper, we address frequency estimation of the signals in the time domain through undersampled data. We analyze the impact of undersampling and demonstrate that three sub-Nyquist channels are generally enough to estimate the frequencies provided the undersampling ratios are pairwise coprime. We deduce the condition that leads to the failure of resolving frequency ambiguity when two coprime undersampling channels are utilized. When three-channel sub-Nyquist samples are used jointly, the frequencies can be determined uniquely and the correct frequencies are estimated. Numerical experiments verify the correctness of our analysis and conclusion.

1. Introduction

Frequency estimation of multiple sinusoids has wide applications in communications, audio, medical instrumentation and electric systems [1–3]. Frequency estimation methods cover classical modified discrete Fourier transform (DFT) [4,5], subspace techniques such as “multiple signal classification” (MUSIC) [6] and “estimating signal parameter via rotational invariance techniques” (ESPRIT) [7] and other advanced spectral estimation approaches [8,9]. In general, the sampling rate of the signal is required to be higher than twice the highest frequency (i.e. the Nyquist rate). The sampling frequency increases as the frequencies of the signals, which results in much hardware cost in applications [10]. In some applications, such as velocity synthetic aperture radar (VSAR) [11], the received signals may be of undersampled nature. So it is necessary to study frequency estimation from undersampled measurements. In addition, this problem has a close connection with phase unwrapping in radar signal processing and sensor networks [12,13].

A number of methods have been proposed to estimate the frequencies with sub-Nyquist sampling. To avoid the frequency ambiguity, Zoltowski proposed a time delay method which requires the time delay difference of the two sampling channels less than or equal to the Nyquist sampling interval [14]. By introducing properly chosen delay lines, and by using sparse linear prediction, the method in [15] provided unambiguous frequency estimates using low A/D conversion rates. Li et al. [16] made use of Chinese remainder theorem (CRT) to overcome the ambiguity problem, but only single frequency determination is considered. Bourdoux used the non-uniform sampling to estimate the frequency [17]. Some scholars used multi-channel sub-Nyquist sampling with different sampling rates to obtain unique signal reconstruction [18,19]. These methods usually impose restriction on the number of the frequency components, which depends on the number of the channels. Based on emerging compressed sensing theory, sub-Nyquist wideband sensing algorithms and corresponding hardware were designed to estimate the power spectrum of a wideband signal [20–22]. However, these methods usually require random samples, which often leads to complicated hardware, making the practicability discounted. In [23] and [24], two channels with coprime undersampling ratios are utilized to estimate line spectra of multiple sinusoids. By considering the difference set of the coprime pair of sample spacings, virtual consecutive samples are generated from second order moments [25]. The method only requires double sub-Nyquist channels without additional processing, the hardware is simpler than the most of former methods.

In this paper, we use three channels other than two channels with coprime undersampling ratios to get enough data. It is demonstrated that the estimated frequencies sometimes can not be uniquely determined when only two channels with coprime undersampling ratios are utilized. In the sampling scheme of multiple channels, if the ambiguous frequencies estimated from single channel are matched successfully, the correct estimated frequencies will be found [26]. Through the analysis for the matching

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process, we deduce the condition that leads to the failure of resolving frequency ambiguity. With the samples obtained from the three channels, the MUSIC algorithm is used to estimate the frequencies, which avoids the complex matching process. The paper is organized as follows: Section 2 gives our analysis and method. Simulation results are shown in Section 3. The last section draws conclusions.

2. Proposed method

2.1. Problem formulation

Consider a complex signal $x(t)$ containing K frequency components with unknown constant amplitudes and phases, and additive noise that is assumed to be a zero-mean stationary complex white Gaussian random process. The samples of the signal at the sampling rate F_S can be written as

$$x(n) = \sum_{k=1}^{K} s_k e^{j(2\pi f_k n/F_S)} + w(n), n = 1, 2, \cdots, \tag{1}$$

where f_k is the kth frequency, s_k is the corresponding complex amplitude, and $w(n)$ is additive Gaussian noise with variance σ^2. Assume that the upper limit of the frequencies is known, but we only have low-rate analog-to-digital converters whose sampling rates are far lower than the Nyquist rate. Undersampling leads to spectral aliasing and frequency ambiguity. Many articles use multi-channel measurement systems to solve the problem. We shall demonstrate that at least three undersampled channels with specific rates can guarantee the success of resolving frequency ambiguity.

2.2. Unfolding in the frequency domain

Suppose the highest frequency contained in the signal is lower than f_H, we sample at the rate $F_{S1} = f_H/a \ (a > 1)$, where a is known as the undersampling ratio.[1] For ease of analysis, a is restricted to be an integer. The collected samples can be written as

$$x(n) = \sum_{k=1}^{K} s_k e^{j2\pi f_k \cdot na/f_H} + w(n), n = 1, 2, \cdots. \tag{2}$$

If we regard these samples as normal data sampled at the Nyquist rate and process them by methods such as DFT or conventional subspace techniques, a formal estimation of $\hat{f}_k$ will be obtained. If $a = 1$, the normalized frequency $\hat{f}_k/f_H$ is the correct estimate. In the case of undersampling, the normalized frequency $\hat{f}_k/f_H$ is actually the estimate of $a \cdot f_k/f_H$, but they can not be one-to-one correspondence because the values of $a \cdot f_k/f_H$ may be greater than 1. In other words, we can not get a unique estimate of f_k from $\hat{f}_k$. Due to the periodicity of trigonometric functions, the estimated normalized frequency $a\hat{f}_k/f_H$ differs from af_k/f_H by an integer κ, i.e.,

$$\hat{f}_k = f_k - \kappa \cdot f_H/a, \kappa \in \mathbb{N}. \tag{3}$$

Without loss of generality, assume that $\hat{f}_k$ is the minimum value that satisfies (3) in the interval $(0, f_H)$, all possible eligible frequencies can be unfolded as

$$\tilde{f}_k = \hat{f}_k + \alpha_k \cdot f_H/a, \ \ \alpha_k = 0, 1, \cdots, a-1. \tag{4}$$

Thus we obtain a series of eligible frequencies from one sub-Nyquist channel. If the true value of α_k is solved, the correct estimate of f_k can be found from $\tilde{f}_k$. Obviously, it's almost impossible to determine the correct frequencies through only one sub-Nyquist sample sequence.

[1] Actually the Nyquist rate in real number field is $2f_H$, in this paper we assume that complex signals can be sampled directly.

2.3. The match of the frequencies

In order to resolve frequency ambiguity caused by undersampling, another channel sampled at the rate $F_{S2} = f_H/b \ (b > 1)$ is required. Consequently, another set of the eligible frequencies can be obtained, namely

$$\tilde{f}'_k = \hat{f}'_k + \beta_k \cdot f_H/b, \ \ \beta_k = 0, 1, \cdots, b-1, \tag{5}$$

where $(*)'$ denotes the parameters related to the second channel. For each k, at least one value of $\tilde{f}'_k$ is the same with some value of $\tilde{f}_k$. In other words, the set composed of $\tilde{f}_k$ and that composed of $\tilde{f}'_k$ contain the same frequency, which is the correct estimate. However, the matchup of the eligible frequencies among different k is unknown. For every k, if $\tilde{f}_k$ and $\tilde{f}'_k$ are matched one to one, the correct frequencies will be found.

To illustrate this process more clearly, we give an example. We assume that the highest frequency in the signal is lower than 60 Hz and the undersampling ratios of the two channels are $a = 3$ and $b = 4$, respectively. The matching process of the eligible frequencies obtained from the two channels are shown in Fig. 1. In Fig. 1(a), the true frequencies are taken as 22 Hz and 25 Hz. Through the first channel, each frequency is unfolded into 3 possible frequencies according to (4). Similarly, 4 possible frequencies are obtained for each true frequency through the second channel. We need to find the equal values in the eligible frequencies of the same frequency component in different channels. We can see that the two sets of eligible frequencies coincide at 22 Hz and 50 Hz, which is the true frequencies. However, such a matching process is not always smooth. In Fig. 1(b), the true frequencies are 25 Hz and 50 Hz. The two sets of eligible frequencies coincide not only at 25 Hz and 50 Hz but also at 5 Hz and 10 Hz. Obviously, matching the eligible values of f_1 with those of f_2 between the two channels results in an erroneous match. In fact, we can not tell which of the matching results is correct unless we know the true frequencies. This matching process also makes sense for more frequency components.

Next we analyze the matching process of the frequencies. Let

$$\tilde{f}_m = \tilde{f}'_l, \ \ l, m = 1, 2, \cdots, K, \tag{6}$$

we have

$$\hat{f}_m + \alpha_m \cdot f_H/a = \hat{f}'_l + \beta_l \cdot f_H/b, \tag{7}$$

i.e.,

$$b\alpha_m - a\beta_l = ab\left(\hat{f}'_l - \hat{f}_m\right)/f_H. \tag{8}$$

The matching process amounts to solving α_m and β_l from (8). Denoting the true values of α_m, β_l by $\bar{\alpha}_m$, $\bar{\beta}_l$, we have

$$f_m = \hat{f}_m + \bar{\alpha}_m \cdot f_H/a, \ \ f_l = \hat{f}'_l + \bar{\beta}_l \cdot f_H/b. \tag{9}$$

Substituting (9) into (8) yields

$$b\alpha_m - a\beta_l = ab(f_l - f_m)/f_H + b\bar{\alpha}_m - a\bar{\beta}_l. \tag{10}$$

To solve the binary indefinite Eq. (10), we introduce the following Bézout's identity [27]:

Theorem 1. *Let a and b be positive integers with greatest common divisor equal to d. Then there are integers u and v such that $au + bv = d$. In addition, the greatest common divisor d is the smallest positive integer that can be written as $au + bv$, and every integer of the form $au + bv$ is a multiple of the greatest common divisor d.*

We focus on the situation that a and b are coprime. According to Bézout's identity, when $a \perp b$, (10) has integer solutions as long as its right hand side is an integer. Moreover, since $\alpha_m \in \{0, 1, \cdots, a-1\}$ and $\beta_l \in \{0, 1, \cdots, b-1\}$, the Eq. (10) just has a unique satisfactory solution. When $l = m$, the unique true values