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Efficient reconstruction of density matrices for high dimensional quantum state tomography

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ABSTRACT

The conventional quantum state tomography (QST) needs large number of measurements to reconstruct the quantum state. Thanks to the compressive sensing (CS) theory, one can recover a pure or nearly pure quantum state with an acceptable accuracy given much fewer number of measurements. However, most existing algorithms for CS based QST are rather slow and difficult to be implemented in practice. To fill the gap between the CS theory and practical QST, this paper firstly applies an improved Alternating Direction Multiplier Method (ADMM) combining with the Iterative Shrinkage-Thresholding Algorithm (ISTA), IST–ADMM for short, aiming at improving the efficiency of QST problem in particular with much lower number of measurements. The IST–ADMM avoids computing the inverse of large-scale matrix, reduces the computational time and required memory space. The computation complexity is reduced from $\mathcal{O}(d^6)$ for least square (widely used in QST), and $\mathcal{O}(md^4)$ for Fixed Point-ADMM in our previous work, to IST–ADMM's $\mathcal{O}(md^2)$. The proposed algorithm makes it practical to reconstruct high dimensional quantum states provided fewer number of measurements. The simulations verify the superiority of the proposed algorithm, where it takes 3.13 minutes to reconstruct an 8-qubit density matrix with 96.17% accuracy, which is faster than many existing and our previous work.

1. Introduction

Quantum state tomography (QST) is a fundamental technique of quantum information processing, and its results determine the accuracy of the system afterwards such as in quantum computing and quantum communication [1]. The process of QST can be described as the process to recover a density matrix ρ of the quantum state given a series of measurements obtained from physical experiments. An n qubits quantum state can be fully represented by its density matrix ρ in a d dimensional Hilbert space [2], where $d = 2^n$ and ρ is a Hermitian matrix. $\mathcal{O}(d^2)$ measurements are usually required to fully recover the density matrix ρ due to the complexity of the system [3].

QST has its unique characteristics that distinguishe it from other optimization problems. Unlike classical systems, quantum mechanical measurements capture collapsed states in probabilities due to the Heisenberg uncertainty principle. Density matrices are supposed to hold its properties, and quantum measurements are costly. The new signal processing approach, compressed sensing (CS), gives us a better alternative to estimate a quantum state with fewer measurements. Proposed by Candes and Donaho [4,5], CS is a novel data acquisition and sampling theory that can process structured signals (sparse, low-rank, etc.) more efficiently [6,7]. It implies that only few essential data is required to exactly reconstruct the original signal by solving a CS-based optimization problem. Fortunately, people are usually interested in pure or nearly pure states practically in quantum systems [8], which means the corresponding density matrix ρ is low-rank and its singular values mostly vanish. In this case, the low-rank density matrix ρ can be reconstructed accurately with fewer measurements. Candes et al. [9] and Cai et al. [10] gave sufficient conditions to precisely reconstruct a k-sparse signal (the signal vector has k non-zero elements). Gross demonstrated that if the sensing matrix A satisfies the rank Restricted Isometry Property (RIP), $\mathcal{O}(d \cdot r \ln d)$ measurements are enough to estimate nearly pure quantum states[1]. However, these works usually provide theoretical bounds with orders, many constants and parameters in the expression, which might confuse researchers with quantum physics backgrounds. Meanwhile, few existing works studied how many measurements are really required in practice, and what the sampling rate is needed for CS based QST

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if we want to recover the density matrix with certain level of accuracy.

From the algorithmic perspective, the algorithms for convex optimization problems via CS have been studied widely. These problems mainly focus on minimizing the ℓ_1 norm or nuclear norm. For minimizing ℓ_1 norm, the algorithms include Interior Point (IP) [11], Gradient Projection (GP) [12] and Iterative Shrinkage-Thresholding Algorithm (ISTA). ISTA was proposed to solve Wavelet image convolution recovery problems [13,14] and later it was widely adopted to solve linear inverse problems with sparse constraints [15]. Daubechies et al. [16] proved the convergence of ISTA. For minimizing the nuclear norm, Yang and Yuan [17] introduced Augmented Lagrangian Method (ALM) and ADMM. Li and Cong [8] firstly applied ADMM to compressive quantum state reconstruction problems with both ℓ_1 norm and nuclear norm. The solution has a good accuracy but the complex computation caused by high dimensional matrix inversion limits its applications. For instance, it takes almost three hours with 92.43% accuracy to reconstruct a 7-qubit density matrix in MATLAB on the computer with 2 cores of 2.4 GHz Intel Xeon E5-2407 CPUs. Hence we hope to improve the algorithm further in terms of efficiency and accuracy.

In this paper, we make several contributions in CS based QST as follows. Firstly, we propose a novel algorithm for density matrix reconstruction given fewer number of measurements, with less computational time and better accuracy. The proposed algorithm combined an adaptive ADMM framework with a nuclear norm ISTA algorithm. Specifically, in each iteration, we use ISTA to obtain the solutions of sub-problems and then use the ADMM to update variables alternately. The IST–ADMM avoids computing the inverse of large-scale matrix, is capable of handling outlier errors, and approaches the true density matrix more rapidly comparing to several prevailing methods and our previous work. In particular, the computation has been reduced from $\mathcal{O}(d^6)$ for least square, $\mathcal{O}(md^4)$ for FP-ADMM in [18], to $\mathcal{O}(md^2)$ for IST–ADMM in this paper. In addition, we estimate the constants and parameters in the theoretical bounds of number of measurements for different qubits derived by previous works. These proper values match our simulation bounds and can provide a guidance to the sampling rates for researchers who intend to implement CS based QST in practice. Moreover, we study the time and accuracy required to reconstruct density matrices of qubits $n = 5, 6, 7, 8$ respectively. Compared with the least square (LS) and previous ADMM method for QST, the proposed algorithm demonstrates its superiority in time and much higher estimation accuracy, especially for high dimensional quantum systems.

This paper is organized as follows. In Section 2, we give the theoretical lower bounds of measurement rates for different qubits using some relative formulas. In Section 3, we propose the IST–ADMM algorithm in detail. Numerical experiments and results analysis are given in Section 4. Finally the conclusion is summarized in Section 5.

2. Lower bounds of measurement rates provided by CS

The state of an n-qubit quantum system can be described by a $d \times d$ density matrix ρ. The task of quantum state tomography is to reconstruct the density matrix ρ given certain measurements. The CS theory provides lower bounds of the number of measurements required to estimate the quantum state. Let $W_1, \cdots, W_{d^2}$ be an orthogonal basis for $\mathbb{C}^{d\times d}$, with respect to the inner product $(W_i, \rho) = tr(W_i^H \rho)$, W_i^H is the Hermitian transpose of W_i. We choose m bases out of d^2 elements, $\omega_1, \cdots, \omega_m$ at random from $\{W_1, \cdots, W_{d^2}\}$. Let the expectation of measurements $b_i \in \mathbb{R}^m$, and measuring operator $\mathcal{A} : \mathbb{C}^{d\times d\to m}$, then $b_i = (\mathcal{A}(\rho))_i + e_i = c \cdot tr(\omega_i^H \rho) + e_i, i = 1, \cdots, m$ or

$$b = A\mathrm{vec}(\rho) + e \tag{1}$$

where vec(·) represents the transformation from a matrix to a vector by stacking the matrix columns; $A \in \mathbb{C}^{m\times d^2}$ is the normalized measurement operator in a matrix form whose ith row consists of the concatenation of ω_i's rows; $e \in \mathbb{R}^m$ represents the noise caused by the system or measuring process; c is a normalized constant. If we set $E(\mathcal{A}^H\mathcal{A}) = \mathcal{I}$ where E represents the expectation overall $\mathcal{A}$, c would be $c = d/\sqrt{m}$ [8], m is the number of measurements.

Since the degree of the freedom of ρ is $d \times d$, usually people need $\mathcal{O}(d^2)$ measurements to give a unique solution by solving a system of linear equations. Yet if the quantum system is known as pure or nearly pure, the density matrix ρ has a low rank [1]. The density matrix reconstruction can be converted to a convex optimization problem by minimizing its nuclear norm [8]:

$$\begin{aligned} &\text{minimize } \|\rho\|_* \\ &\text{s.t. } \|A\mathrm{vec}(\rho) - b\|_2^2 \le \varepsilon, \rho^H = \rho, \rho \succeq 0 \end{aligned} \tag{2}$$

where $\rho \in \mathbb{C}^{d\times d}$; $\|\cdot\|_*$ represents the nuclear norm, which is equal to the sum of singular values, $m \ll d^2$; $\|\cdot\|_2$ represents the ℓ_2 norm, ρ^H is the Hermitian transpose of ρ; $\rho \succeq 0$ means ρ is a positive semi-definite matrix.

To make sure that the fewer measurements b contain the essential information of ρ, A is supposed to satisfy the rank RIP [19]. The lower bound of measurements has been derived according to RIP [20]. The lower bound implies that how many measurements do one need to exactly reconstruct an unknown low-rank matrix ρ given A's property. In general, there are several types of matrices that can be used as a sensing matrix A [21], such as Gaussian random matrix and Bernouli random matrix etc. Here we choose Pauli matrices in this paper as the bases of A, because they are widely used and easy to be implemented in practical quantum measurement.

The Pauli bases, $\omega_i = \otimes_1^n \sigma_k$, $k \in 1, \cdots, 4$, for an n-qubit quantum system are the Kronecker product of a series of complex and unitary elemental 2×2 Pauli matrices σ_i chosen from the four possibilities randomly:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus there are $(2^n)^2 = d^2$ Pauli bases totally. We choose m Pauli bases randomly and record them as $\omega_{a_1}, \cdots, \omega_{a_m}$, where $a_1, \cdots, a_m \in [1, d^2]$, then the sensing matrix A is:

$$A = \begin{pmatrix} \mathrm{vec}(\omega(a_1))^T \\ \vdots \\ \mathrm{vec}(\omega(a_m))^T \end{pmatrix}. \tag{3}$$

In the quantum state tomography, we define the measurement rate η as

$$\eta = m/d^2. \tag{4}$$

The lower bound of η has been studied when Pauli bases are used as the sensing matrix and $\mathrm{rank}(\rho) = r$. If η satisfies [20]

$$\eta \ge C(1+\beta) r \ln d/d, \tag{5}$$

the solution ρ^* to the optimization problem (2) is unique and equals ρ with the overwhelming probability

$$P_s \ge 1 - e^{-\beta}, \tag{6}$$

where $\beta > 0$ is a parameter that balances the lower bound of η and the probability P_s. Some conclusions can be drawn from the