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Spline-based finite-element method for the stationary quasi-geostrophic equations on arbitrary shaped coastal boundaries

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Abstract

This work concerns a B-spline based finite-element algorithm for the stationary quasi-geostrophic equations to treat the large scale wind-driven ocean circulation on arbitrary shaped domains. The algorithm models arbitrary shaped coastal boundaries on intra-element, or embedded boundaries. Dirichlet boundary conditions on the embedded boundaries are weakly imposed and stabilization is achieved via Nitsche's method. We employ a hierarchical local refinement approach to improve the geometrical representation of curved boundaries. Results from several benchmark problems on rectangular and curved domains are provided to demonstrate the accuracy and robustness of the method. We also provide the Mediterranean sea example that illustrates the effectiveness of the approach in the wind-driven ocean circulation simulation.

Keywords: Nitsche's method; Geophysical fluid dynamics; Ocean circulation; Fourth-order partial differential equations

1. Introduction

The quasi-geostrophic equations (QGE) are one of the standard mathematical models for large scale oceanic and atmospheric flows (Vallis [1], Cushman-Roisin and Beckers [2], Majda [3], Majda and Wang [4], Pedlosky [5], McWilliams [6]). They are often used to model the dynamics of climate (Dijkstra [7]). As a simplified model, the QGE allow for efficient computational simulations while preserving many of the essential features of the underlying large scale ocean flows such as strong western boundary currents, weak eastern boundary currents, and weak interior flows. Of particular interest to this study is the spline-based finite-element method for the streamfunction formulation of the QGE, allowing for better modeling of the large scale wind-driven ocean circulation on irregular geometries. Accurate modeling features like coastlines are important in ocean models. This is because numerical artifacts from stepwise boundaries can affect ocean circulation predictions over long time integration (Adcroft and Marshall [8], Dupont et al. [9], and Wang et al. [10]).

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Most finite-element methods of the QGE have been developed based on the mixed streamfunction-vorticity formulation rather than the pure streamfunction formulation. This is because the streamfunction-vorticity formulation is a second-order partial differential equation (PDE). Thus, its conforming finite-element discretization requires easy-to-implement low-order C^0-elements. To the best of our knowledge, its available error estimates are suboptimal (Fix [11]). This study focuses on the pure streamfunction formulation with the optimal error estimates (Foster et al. [12]). The streamfunction formulation is, however, the fourth-order PDE which requires higher-order C^1-elements that make implementation challenging. Recently, Kim et al. [13] developed a finite-element method using B-spline basis functions to discretize the streamfunction formulation of the stationary quasi-geostrophic equations (SQGE) and two simplified versions of that model, namely the linear Stommel and Stommel–Munk models. They showed the optimal rate of convergence for several benchmark problems on a rectangular domain.

The main purpose of this paper is to extend their work to deal with nonrectangular domains to model the large scale ocean circulation with arbitrarily shaped coastal boundaries. To achieve this goal, we develop a B-spline based finite-element algorithm capable of modeling arbitrary shaped complex boundary embedded within the background mesh. The boundary conditions on the embedded boundary are weakly enforced based on the idea of Nitsche's method [14]. Recently, Nitsche's method has been successfully applied for weakly imposing boundary and interfacial conditions in standard finite-element methods for fourth-order PDEs (Kim et al. [15], Kim and Dolbow [16], Kim et al. [17], and Kim et al. [18]) and embedded finite-element methods (Hansbo and Hansbo [19] and Dolbow and Harari [20]). Moreover, it has also been used with success for weakly imposing boundary (Embar et al. [21], Embar et al. [22], and Kim et al. [23]) and interfacial conditions (Jiang et al. [24]) in a spline-based finite-element method.

Our embedded approach is closely related to the finite cell method (Reuss et al. [25]). However, these two approaches originate from different concepts. While our approach has its origin from the extended finite-element method with Heaviside enrichments for voids (Sukumar et al. [26]), the finite cell method embeds an arbitrary complex geometry based on the idea of the fictitious domain. In contrast to our method using Heaviside enrichments, the finite cell method penalizes the elasticity tensor by a very small parameter to confine the influence of a nonphysical domain. The finite cell method has been successfully implemented and tested for many second-order elasticity problems with complex domains. The recent work in the finite cell method (Kollmannsberger et al. [27]) proposed a parameter-free formulation to weakly enforce essential boundary conditions. Their formulations are, however, only developed for the second-order PDE problems, not for the fourth-order PDEs considered in this study.

With embedded approaches, the boundary is often implicitly represented by a level set function. The level of accuracy in the geometric representation of the boundary becomes important when higher-order rates of convergence are sought. To improve the geometrical representation of the curved boundary, we employ a h-refinement approach to locally refine the mesh in the vicinity of the boundary. The level set function is interpolated by means of finite-element shape functions, which need not necessarily to be the same ones used for the finite-element approximation. This allows one to introduce an adaptive mesh to describe the geometry while keeping a higher-order approximation of field quantities on a uniform coarse mesh. With this approach, an optimal rate of convergence with a higher-order extended finite-element method (XFEM) was obtained for embedded interface problems (Dréau et al. [28], Legrain et al. [29] and Jiang et al. [24]). The similar techniques were also used in the finite cell method to improve the accuracy of numerical integration on cut cells (Schillinger et al. [30]). In this study, we employ similar techniques to represent the level set on a refined sub-mesh.

The paper is organized as follows. In Section 2, we briefly present the Nitsche-type variational formulations, introduced by Kim et al. [13], for the SQGE as well as linear Stommel and Stommel–Munk models. In Section 3, the finite-element discretization with B-splines is briefly explained. In Section 4, we introduce the hierarchical local refinement approach to model accurately the curved boundary based on the level set. In Section 5, numerical studies with embedded boundaries are performed for these Nitsche-type formulations. In Section 6, we finally provide conclusions and a summary of directions for future work.

2. Model problems with Nitsche-type variational formulations

In this section, we present the governing equations of the SQGE, the Stommel–Munk model, and the Stommel model as well as their Nitsche-type variational formulations introduced by Kim et al. [13]. Unless otherwise indicated, we assume that all admissible streamfunctions and test fields belong to the Sobolev space $\mathcal{S} = H^2(\Omega)$ of functions