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# PDF approximations to estimation and detection in time-correlated alpha-stable channels

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ABSTRACT

This paper considers the estimation and detection problems for statistically dependent heavy-tailed signals with no closed-form probability density function (PDF). We propose two parametric PDF approximations for symmetric $\alpha$-stable (S$\alpha$S) distribution to be utilized in approaches based on the Maximum likelihood (ML) criterion. The nonlinear least square (LS) and curve fitting are used to compute parameters of the new formulations which are functions of the characteristic exponent. Moreover, we study binary signal detection in channels with time-dependent heavy-tailed noise modeled by S$\alpha$S distribution and first order autoregressive (AR(1)) process. Using the novel PDF approximations in the ML estimator, an algorithm for model parameters estimation of the noise is initially developed. Then, new suboptimal receivers are designed through the use of the new PDF formulations and parameter estimates. Numerical results demonstrate the superiority of the proposed approximations over the existing formulations, and also good accuracy for the estimation algorithm. Additionally, it is shown that the proposed detectors operate near optimal receiver and also outperform the other suboptimal detectors, especially when $\alpha$ is small.

## 1. Introduction

The impulsive behavior of many natural and man-made noise sources can be suitably described by symmetric $\alpha$-stable (S$\alpha$S) distribution [1]. For instance, shot noise, radar clutter [2,3], co-channel interference in some wireless networks [4], and also the noises that appear in underwater channels and power line communication [5,6] are properly modeled by S$\alpha$S distribution. It is usually assumed that the noise process has statistically independent samples. Nevertheless, different studies have shown that in several situations, the additive noise samples are statistically dependent in the time-domain [7]. For example, the independent heavy tailed noise that passed through the narrow band filter presents a dependent structure [8]. As another example, in smart grid monitoring tasks, the measured noise sequence in the power substation is presented by a hidden Markov model and heavy tailed distribution [9]. Furthermore, the authors in [10] have studied signal detection in the first order moving average noise. Also, the classification of the digital amplitude-phase modulated signal has been considered in heavy-tailed autoregressive noise [7]. It is noteworthy that the performance of a detector derived under independent noise assumption, significantly degrades in situations where the noise is time-dependent.

In the signal processing applications for signals with heavy-tailed distributions, there is no closed-form expression for S$\alpha$S PDF, but, it is defined according to its characteristic function (CF). Consequently, the estimation and detection tasks based on the ML criterion need numerical integrations. This significantly imposes computational complexity and implementation difficulty [11,12]. To deal with this problem, an alternate model for the S$\alpha$S PDF can be employed. The mixture models are used to suitably approximate the S$\alpha$S distribution. In [13–15] three types of PDF approximation using the mixture models have been introduced, namely, the Gaussian mixture model, mixture of Gaussian and Cauchy and mixture of Cauchy. The mixture of Gaussian and Cauchy model fits the S$\alpha$S distribution better than the Gaussian mixture model, but the mixture of Cauchy model outperforms the other ones. Nevertheless, these approximations can not properly capture the tail of the $\alpha$-stable distribution. In [16], using the asymptotic heavy tail behavior of the S$\alpha$S PDF, power-based approximation has been employed to capture the decay rate of the distribution. For signal detection in the presence of S$\alpha$S noise, due to the lack of closed-form expression for PDF, several suboptimal detectors have appeared in the literature [17,2,18,19]. Gaussian detector as a linear receiver gives significantly poor performance in a very impulsive environment. The Cauchy receiver with better performance and more complexity, is optimal when $\alpha = 1$. The myriad detector based on the Cauchy distribution has been proposed to improve the performance of sub-

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optimal detectors over a wide range of $\alpha$ [20]. Recently, the suboptimal detectors using the alternate PDF approximations based on the mixture of Cauchy and power-based function, have been introduced to get better performance especially for small values of $\alpha$ [15,16].

In this paper, we approximate the S$\alpha$S distribution by two parametric formulations using a rational function with fractional power series and also a multiplication of exponential and rational functions. For each approximation, the non-linear LS algorithm is employed to estimate the model parameters which are functions of the characteristic exponent $\alpha$. Using the polynomial fitting and spline theory for these model parameters, we can find an approximated function of S$\alpha$S PDF for every arbitrary value of $\alpha$. Numerical results show that the proposed models have good accuracy and outperform the existing approximations, in terms of integrated square error and Jensen-Shannon divergence. Moreover, we investigate binary signal detection in the presence of time-dependent heavy-tailed noise described by the $\alpha$-stable AR(1) model. Based on the new approximations, we firstly propose an algorithm to estimate the model parameters of noise, and then develop new detectors by considering the dependency structure of noise. Numerical results present good accuracy of estimates and the outperforming performance of the new detectors compared to commonly used suboptimal detectors. Furthermore, the comparison of decision regions of the new detectors with the optimal one reveals the near optimal performance of the proposed receivers.

The remaining sections are organized as follows. The time-dependent impulsive noise model is described in Section 2. We propose two parametric approximation models for the S$\alpha$S distribution in Section 3. An algorithm for model parameters estimation of noise is proposed in Section 4. In Section 5, the suboptimum receiver structures are derived and numerically compared through decision boundaries. Finally, using the Monte Carlo simulation, the performance of the proposed algorithms is evaluated in Section 6.

## 2. Noise model

In most radio environments, the additive disturbance signal is non-Gaussian and statistically dependent in time. Furthermore, AR modeling has been frequently used to depict the correlation structure by a few parameters [21,22]. In this paper, we assume that the additive noise is a S$\alpha$S process with the first order autoregressive model as following,

$$x[n] = \xi x[n-1] + z[n], \quad z[n] \sim S\alpha S(\alpha, \rho). \tag{1}$$

where, $\xi$ denotes the autoregressive coefficient and $z[n]$ indicates the excitation process with S$\alpha$S distribution. The $\alpha$-stale distribution does not usually have closed-form expression for the PDF and is completely characterized by its CF as $\Phi(\omega) = \exp\{\Psi(\omega)\}$ when,

$$\Psi(\omega) = \begin{cases} j\mu\omega - \rho^\alpha|\omega|^\alpha\{1 + j\beta\,\mathrm{sgn}(\omega)\tan(\alpha\pi/2)\} & \text{if } \alpha \neq 1 \\ j\mu\omega - \rho|\omega|\{1 + j\frac{2}{\pi}\beta\,\mathrm{sgn}(\omega)\log|\omega|\} & \text{if } \alpha = 1 \end{cases} \tag{2}$$

where, $\alpha \in (0, 2]$ is the characteristic exponent which determines the tail heaviness and the decay rate of the distribution. A smaller value of $\alpha$ provides heavier tail for the distribution, but, a value of $\alpha$ near 2 shows more Gaussian behavior. The noise dispersion $\rho$ is a positive constant that indicates the spread of the distribution around its center. The shift parameter $\mu$ determines the mean of the distribution when $1 < \alpha \leq 2$ and its median when $0 < \alpha \leq 1$. The skewness parameter $\beta$ exhibits the symmetry of the distribution around its location parameter. Among the class of $\alpha$-stable distributions, only the Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1$) and Lévy ($\alpha = 0.5,\ \beta = 0$) distributions provide the closed-form expressions. In particular, by setting $\mu = 0$ and $\beta = 0$, the CF of the S$\alpha$S $(\alpha, \rho)$ has the form of,

$$\Phi(\omega) = e^{-\rho^\alpha|\omega|^\alpha}, \tag{3}$$

Using the inverse Fourier transform, the S$\alpha$S PDF $f_\alpha(z)$ can be numerically calculated. However, in applications based on the PDF, the numerical evaluation of the integration takes a long time, especially for smaller values of $\alpha$. To circumvent this problem, we propose PDF approximations to the S$\alpha$S distribution that can be employed in estimation and detection problems.

## 3. Approximations to the S$\alpha$S distribution

As explained before, to approximate the S$\alpha$S PDF $f_\alpha(z)$, some alternate formulations have been proposed. Here, before investigating our proposed models, we briefly review three recent approximations to the S$\alpha$S distribution.

A mixture of Cauchy and Gaussian distributions (MoCG) has been frequently used to present an alternative PDF formulation as [23],

$$\hat{f}_\alpha(z) = \frac{k\rho}{\pi(\rho^2 + z^2)} + (1-k)\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left(-\frac{z^2}{2\sigma_g^2}\right) \tag{4}$$

where $k$ is a weighting factor that depends on $\alpha$, also, $\sigma_g$ is the Gaussian variance and $\rho$ is the Cauchy scale parameter. In the literature [24], it has been shown that the S$\alpha$S PDF decays proportional to $z^{-(1+\alpha)}$. Using this fact, another alternate S$\alpha$S PDF has been proposed according to power function as [16],

$$\hat{f}_\alpha(z) = \frac{k_1}{1 + k_2|z|^{\alpha+1}} \tag{5}$$

The parameters of $k_1$ and $k_2$ can be estimated. The expression in (5) captures asymptotic decaying rate of the S$\alpha$S distribution. Recently, another alternate PDF formulation for the S$\alpha$S distribution has been used as a mixture of two Cauchy distributions [15],

$$\hat{f}_\alpha(z) = \frac{k_1}{1 + k_2 z^2} + \frac{k_3}{1 + k_4 z^2} \tag{6}$$

It is required to estimate the unknown parameters of $k_1$, $k_2$, $k_3$ and $k_4$. In order to get a more general approximation, the two terms in (6) have been combined to convert to a rational function [15]. In the following, we propose two parametric expressions as new alternate models for the S$\alpha$S $(\alpha, \rho)$ PDF. Without loss of generality in these approximations, it is assumed that the scale parameter is $\rho_z = 1$.

### 3.1. Rational function

Here, we address the S$\alpha$S PDF approximation through the technique of mixture modeling [25] by utilizing the long-tailed distributions to form a non-Gaussian mixture model (e.g. mixture of two Cauchy distributions [15]). In this regard, we use the power-based functions as heavy-tailed components in the mixture model to explicitly improve the accuracy of S$\alpha$S PDF approximation as following,

$$\hat{f}_\alpha(z) = \frac{\acute{k}_1}{1 + \acute{k}_2|z|^{\alpha+1}} + \frac{\acute{k}_3}{1 + \acute{k}_4|z|^{\alpha+1}} \tag{7}$$

According to [15], to have a more general and accurate expression, we can combine the two components in (7) to form a rational function of fractional power series as the following expression,

$$\hat{f}_\alpha(z) = \frac{k_1 + k_2|z|^{\alpha+1}}{1 + k_3|z|^{\alpha+1} + k_4|z|^{2(\alpha+1)}} \tag{8}$$

Due to the fractional power series, this approximation is more complicated than the mixture of two Cauchy distributions. There are different methods to determine the model parameters values of $k_1,\ k_2,\ k_3$ and $k_4$, which are functions of $\alpha$. We use the nonlinear LS estimator to determine these unknown model parameters values as follows,

$$\underset{\{k_1,k_2,k_3,k_4\}}{\operatorname{argmin}} \int |f_\alpha(z) - \hat{f}_\alpha(z)|^2 dz \tag{9}$$

where, $f_\alpha(z)$ denotes the true S$\alpha$S PDF and $\hat{f}_\alpha(z)$ is expressed in (8). In