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Theoretical background and implementation of the finite element method for multi-dimensional Fokker–Planck equation analysis

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ABSTRACT

Fokker–Planck equation is one of the most important tools for investigation of dynamic systems under random excitation. Finite Element Method represents very effective solution possibility particularly when transition processes are investigated or more detailed solution is needed. However, a number of specific problems must be overcome. They follow predominantly from the large multi-dimensionality of the Fokker–Planck equation, shape of the definition domain and usual requirements on the nature of the solution which are out of a conventional practice of the Finite Element employment. Unlike earlier studies it is coming to light that multi-dimensional simplex elements are the most suitable to be deployed. Moreover, new original algorithms for the multi-dimensional mesh generating were developed as well as original procedure of the governing differential and algebraic systems assembling and subsequent analysis. Finally, an illustrative example is presented together with aspects typical for the problem with large multi-dimensionality.

1. Introduction

The Fokker-Planck equation (FPE) is a frequently used tool for solution of cross probability density function (PDF) of a dynamic system excited by a vector of random processes of Wiener type. Input random processes are commonly considered as Gaussian white noises being stationary or non-stationary. They affect the system as additive or multiplicative processes. A large number of monographs and papers have been published concerning this widely known partial differential equation (PDE) which has mostly the linear character, although more complex definitions exist as well. For comprehensive explanation see, for instance, [1–5]. For completeness let us specify differential system considered, which leads to the commonly used FPE for investigation of the cross probability density function (PDF) of the system response:

$$\frac{\mathrm{d}x_j(t)}{\mathrm{d}t} = f_j(\mathbf{x},t) + g_{jr}(\mathbf{x},t)w_r(t)\,, \quad j = 1,\ldots,2n,$$
$$n - \text{dynamic degrees of freedom} \tag{1}$$

- $\mathbf{x} = [x_1, x_2, .., x_{2n}]^T$ - response components (hereafter space variables): (i) x_{2j-1} - displacements; (ii) x_{2j} - velocities,
- $w_r(t)$ - Gaussian white noises with constant cross-density in a meaning of stochastic moments $K_{rs} = \mathcal{E}\{w_r \cdot w_s\};\ r, s = 1, m,\ m -$ number of acting noises
- $\mathcal{E}\{\cdot\}$ - mathematical mean value operator in the Gaussian meaning,
- $f_j(\mathbf{x},\ t)$, $g_{jr}(\mathbf{x},\ t)$ - continuous deterministic functions of state variables $\mathbf{x}$ and time t; $j = 1, 2n$.

If input processes w_j, w_r can be considered to be Gaussian white noises, the respective FPE for an unknown PDF in variables $\mathbf{x}$, t can be associated to Eq. (1):

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial x_j}\Big(\kappa_j(\mathbf{x},t)\cdot p(\mathbf{x},t)\Big) + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_k}\left(\kappa_{jk}(\mathbf{x},t)\cdot p(\mathbf{x},t)\right) \tag{2}$$

$$\kappa_j(\mathbf{x},t) = f_j(\mathbf{x},t) + \frac{1}{2}K_{rs}\cdot g_{ls}(\mathbf{x},t)\frac{\partial g_{jr}(\mathbf{x},t)}{\partial x_l};$$
$$\kappa_{jk}(\mathbf{x},t) = K_{rs}\cdot g_{jr}(\mathbf{x},t)g_{ks}(\mathbf{x},t) \tag{3}$$

Abbreviations: ADES, Advances in Engineering Software (journal name); CPU, Central Processing Unit (time consumption); DOF, Degree of freedom; FDM, Finite Difference Method,; FEM, Finite Element Method; FE, Finite Element (used when a particular element is addressed); FPE, Fokker–sPlanck equation; JPEM, Journal of Probabilistic Engineering Mechanics; MCD, Method of Central Differences; MD, multi-dimensional (space); MDOF, n-DOF, Multiple degree of freedom or n-Degree of freedom; ODE, ordinary differential equation (system); PDE, partial differential equation; PDF, probability density function; SDOF, TDOF, Single or Two degrees of freedom (system); 2D, 3D, 4D, two, three or four dimensional (space or problem); SM, EM, System matrix or Evolution matrix; SPMD, Single Program-Multiple Data parallel computing.

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$\kappa_j(\mathbf{x}, t)$ - drift coefficients; $\kappa_{jk}(\mathbf{x}, t)$ - diffusion coefficients.

An overview of analytical and semi-analytical ways of FPE is given in various resources. One of the most important monographs of this type seems to be [6]. Many methods are outlined also in basic monographs cited above. A number of special oriented papers appeared dealing with analytical or partly analytical FPE solution procedures, e.g. [7,8].

Let us make a brief remark about some of them although boundaries between relevant groups are rather blured. Very widely used particularly in theoretical physics are Fourier decomposition based procedures. Being based on a separability $p(\mathbf{x},t) = p(\mathbf{x}) \cdot \varphi(t)$, drift and diffusion coefficients should be time independent. Boltzmann type solution is frequently used as a basic step for subsequent analysis employing perturbation techniques. Galerkin–Petrov method is probably the most general and suitable for investigation for majority of tasks formulated in terms of FPE. Many other special case oriented methods are applicable being based on idea of completion of various potentials, first integrals and their combinations, free parameters fitting, etc. Besides indisputable strengths of these methods and excellent results brought by them, they include very unpleasant shortcomings which consist predominantly in very limited dimensionality of FPE, possibilities in configuration of boundary conditions and problematic possibility to analyze any non-stationary problem.

Numerical methods applied to analysis of FPE appeared a bit later due to high demands on the CPU performance. The first attempts at the FEM application in numerical treatment of the FPE date back to the early seventies. As the first systematic studies devoted to Galerkin-Petrov type formulation of FEM can be considered publications by Bergman and Heinrich [9]. A series of further papers followed, see, for instance, [10–13]. The paper [14] reminded the Czech community the FEM variant of FPE numerical solution possibility. Many authors have been dealing with various aspects of special FEM variants related with the Galerkin method applied to FPE. Anyway, FPE is not self-adjoint and therefore variational methods based on orthogonalization principles should be employed. A stationary solution has been discussed, for instance, in [15,16], for a multi-scale version, see among others [17], etc. Some more knowledge at the field contributed also authors of this paper [18]. Regarding numerical procedures, the comprehensive state of the art concerning applications of numerical methods for analysis of the FPE has been published in JPEM by a team of twenty authors under editorship of Schuëller, see [19]. Another review focused to FEM has been published by Johnson at al. [20].

Several papers appeared during a couple of the last years using Finite Difference Method (FDM) to solve FPE associated with SDOF and TDOF systems of Duffing type, see e.g. [21,22]. Some results presented in these papers are very close to those presented in Section 5.1 of this paper. However, the FDM has not been taken into consideration by authors of this study due to well-known shortcomings of FDM and other numerical methods in comparison with FEM. It applies particularly Finite Elements (FE) formulated on the Galerkin–Petrov basis, see [9,12,13]. See also numerous famous monographs and papers dealing either with FEM or with numerical methods in general being written from strictly mathematical point of view as well as literature concerning very practical applications. They prefer FEM all of them.

Let us mention for completeness a couple of special methods which are less sensitive regarding the DOF increase. They are based on Bolzmann type solutions and subsequent multiscale based principle extrapolation. Another way works with the Maximum Entropy of probability density principle, see e.g. [23,24]. As the next let us refer [25], where stochastic modelling of gene regulatory networks are studied to understand how random events at the molecular level influence cellular functions. Another procedure represents the semi-implicit integration factor method for high-dimensional systems which has been introduced in [26]. It indicates potential broad applications of the sparse grid technique when treating PDE in high-dimensional spaces with cross-derivatives and non-constant diffusion coefficients. All these methods, however, concern only very narrow class of special problems disposing with similar nonlinear normal modes (without local modes and homoclinic orbits), stochastically proportional systems, etc. They do not enable a generalization as needed and usually suffer from unpleasant problems of numerical instability during the phase of parameter fitting. The primary motivation of our work, however, comes from area of aero-elasticity, stochastic stability and traffic engineering, where these special approaches cannot be applied and the general methods must be involved.

It should be emphasized that an unpleasant factor of the FPE space variables number is increasing with rising DOF of the system investigated. This problem is general and significantly restricting whatever method (analytical/numerical) for its solution or even partial investigation is considered. After all it concerns more or less also other methods commonly used in stochastic dynamics avoiding FPE or other variants of Markov process application. This factor is widely commented in [4] and mainly in [6] concerning semi-analytical approaches with some more or less heuristic rough estimates of upper (rather not too high) limits applicable.

Although numerical approaches and namely FEM application provided many results inaccessible by analytical ways, some obstacles have been encountered as well. Papers published until now using FEM for the FPE analysis are dealing with single degree of freedom (SDOF) systems only, despite also other than Gaussian inputs are discussed. So the respective FPE includes two independent space variables only (x_1, x_2). However, the aim of this study is to deal with multiple degree of freedom (MDOF) systems ($n \geq 2$). Regarding FEM, to the best knowledge of the authors, this paper is the first attempt for implementation of the FEM to solve FPE for Gaussian random noises, when DOF$\geq$ 2. It means that the system Eq. (1) includes $2n$ space variables or in other words that the vector $\mathbf{x}$ in respective FPE, Eq. (2), includes $2n$ independent space variables. It means for instance 12 space variables when random motion of a rigid body in space with six degrees of freedom is studied. It is quite a high number taking into account that the usual number of space variables in conventional problems of FEM analysis applied in continuum mechanics is two or three. However, the Multi-Dimensional (MD) character of FPE implies the need to reconsider all principle steps to be done on the way of solution by means of FEM. Many requirements should be respected which are out of a conventional practice of the Finite Element employment.

Some steps including the FEM analysis which will be presented in subsequent sections enable doubtlessly further partial improvement and acceleration (mesh generation and polishing, matrices generation, ODE system solution, stability concurrent checking, etc.). Anyway, rough assumptions in the contemporary time appear 6–8 DOF, i.e. 12–16 space variables in the FPE, when extrapolating actual needs of individual steps which must be performed including the overhead costs (PC - four cores CPU). For the near future it seems that the upper limit of performance of numerical methods and particularly of FEM as the FPE investigation tool will rise with power of super-computers available. For some initial contemplations, see e.g. [20]. Authors started already relevant activities oriented towards this challenging strategy.

2. Multi-dimensional finite element

The study concentrates to basic formulation of FPE in the form Eq. (2) admitting Gaussian inputs only, although the range of problems solvable by FPE is much wider (as shows literature dealing with non-Gaussian input process, non-synchronous processes re-