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Numerical Laplace inversion in problems of elastodynamics: Comparison of four algorithms

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ABSTRACT

The objective of this work is to find a suitable algorithm for numerical Laplace inversion which could be used for effective and precise solution of elastodynamic problems. For this purpose, the capabilities of four algorithms are studied using three transforms resulted from analytical solutions of longitudinal waves in a thin rod, flexural waves in a thin beam and plane waves in a strip. In particular, the Gaver–Stehfest algorithm, the Gaver–Wynn's rho algorithm, the Fixed-Talbot algorithm and the FFT algorithm combined with Wynn's epsilon accelerator are tested. The codes written in Maple 16 employing multi-precision computations are presented for each method. Given the results obtained, the last mentioned algorithm proves to be the best. It is most efficient and it gives results of reasonable accuracy nearly for all tested times ranging from 3×10^{-7} s to 3×10^{3} s.

1. Introduction

In many science and engineering applications, integral transforms represent an important tool for solving ordinary/partial differential and integral equations of both integer and non-integer orders [1,2]. The main reason of this fact is that the solution can be found more easily in a transformed domain than in the original one. Laplace transform is one of the most frequently used and its inversion (ILT) is then an important step of the solving procedure. The inversion can be performed analytically using its definition or by the use of existing extensive Laplace transform tables [3–5] only in a limited number of problems. But in many practical applications the transforms are not given by closed analytical formulas and hence numerical inverse Laplace transform (NILT) has to be performed. Unfortunately, the numerical inversion often becomes ill-posed, see [6].

As stated in [7], more than one hundred algorithms for NILT exists. They can be divided into several groups according to the principles which are based on. Probably most of them make the use of the series expansion, usually power series or expansions in terms of orthogonal polynomials are used. The latter type is characterised by better convergence properties and number of works involving Chebyshev, Legendre or Laguerre polynomials can be found [8]. Weeks method, based on orthonormal Laguerre functions expansion [9], is the best-known and it has been modified and improved by many authors (e.g. [10–12]), mainly in the sense of the determination of two free parameters [13]. Despite of this fact, this method is not frequently used nowadays. Quite different situation occurs with methods based on the approximation of Bromwich integral defining ILT [2]. The detailed discussion of such methods can be found in an excellent book [8]. Most of them convert the problem of Bromwich integral to the problem of Fourier integral and usually utilise Fourier series for its evaluation. In addition, the process of inversion can be significantly speed up by the use of FFT in this case [14]. Number of relevant references can be found in [7].

Other algorithms which passed the test of time are based on the well-known Post–Widder (P–W) formula [15,16]. The main disadvantages of these methods consist in slow convergence of resulting sequences and in difficult numerical computation of high-order derivatives. The earlier problem is overcome by the existence of extrapolation methods [17]. The second one is partially resolved by the use of powerful Computer Algebra Systems (CASs) as Maple, Mathematica etc. The usage of so called Gaver functionals which represent the discrete analog of P–W formula is another possibility how to avoid the latter difficulty. It was firstly published by Gaver [18] and as stated in [19] this approach gives better results than the methods based on classical P–W formula and gives good accuracy for a wide range of functions.

The deformation of Bromwich integration contour is another popular approach to ILT. This idea was first published by Talbot

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[20] and as mentioned in [7] only a few algorithms have been developed. The main drawback of this method is the need of complex calculus, but on the other hand when a multi-precision implementation in some CAS is used, it results in a quite simple procedure [7]. There exist some modifications of the original Talbot's algorithm, see e.g. [7,21,22], which mainly differ in the choice of integration path and used integration rule.

Based on the previous description of possible NILT approaches it is obvious that nearly all methods involve the problem of infinite series (or sequences of partial sums), usually with low rate of convergence. To speed up the summation process, extrapolation methods can be used. Such methods transform the original sequence into a different sequence which converges to the same limit but with better rate of convergence. The history of sequence accelerators is mapped in detail in Brezinski's book [17]. In general, linear and non-linear sequence transformations exist. Salzer summation [23] is probably the best-known linear sequence transformation. In the well-known Gaver–Stehfest method [24] this algorithm is used to accelerate the logarithmically convergent sequence of Gaver functionals. Non-linear sequence transformations, like Wynn's rho [25] and epsilon [26] algorithm, Brezinski's theta algorithm [27] or quotient-difference algorithm [28], have greater computer demands than the linear ones, but they are more powerful and succeed where the linear methods fail. That is why these non-regular algorithms are very popular, as declared by many references stated in [29]. Unfortunately, there exists no universal transformation to accelerate arbitrary sequence, as demonstrated by Sidi in [30] or as proved for logarithmically convergent sequences by Delahaye and Germain-Bonne in [31].

Many authors deal with the analysis and testing of different NILT algorithms with appropriate sequence accelerators. We can mention the former comparative studies of Davies and Martin [19] and Duffy's paper [32] or more recent papers of Abate and Valkó [7,33], Valkó and Vajda [34], Hassanzadeh and Pooladi-Darvish [35] or Sheng et al. [36]. These authors test the methods using let say standardised set of transforms or transforms resulting from practical applications. Moreover, some of them use the multi-precision implementations in CASs, namely in Mathematica, and show that under certain conditions results of arbitrary precision can be obtained. But all these works point out that the right choice of the combination of a NILT method and a sequence accelerator depends on the problem solved and the use of at least two different algorithms is recommended to obtained relevant and accurate results.

The purpose of this paper is to find a suitable and effective ILT algorithm for numerical inversion of transforms arising in 1D and 2D problems of wave propagation in solids. Using the analytical solution of longitudinal waves in a thin elastic rod [37], flexural waves in a simply supported thin Timoshenko viscoelastic beam [38] and the analytical solution of plane waves in a thin viscoelastic strip [39] we will study capabilities, accuracy and efficiency of four NILT algorithms which are based on: the Gaver functionals and Salzer summation [33], the Gaver functionals and Wynn's rho algorithm [7], fixed Talbot algorithm [7], FFT and Wynn's epsilon algorithm [40] (FFTe). Contrary to other works in which the inversion is tested either for extremely short times (e.g. of orders 10^{-8}s, see [40]) or for times of orders $10^0 - 10^2$s (e.g. see [7,34]), this work will study the capabilities of all above mentioned methods for times of orders $10^{-7} - 10^3$s, such that both extreme values of times will be taken into account. The testing will be performed by the help of multi-precision computing provided by Maple 16. This approach enables us to determine the dependence of accuracy (the number of significant digits) on the number of decimal digits of precision for all times studied which is important information for the application of such numerical procedures.

2. Laplace transform and description of used test transforms

As stated in [8], the most familiar formula for the expression of the Laplace transform $F(p)$, $p \in \mathrm{C}$, of a real function $f(t)$ was given by Doetsch as

$$F(p) = \int_0^\infty e^{-pt} f(t) dt. \tag{1}$$

It is assumed that the function $f(t)$ is defined for all $t \in \langle 0, \infty)$ and that the integral in (1) is convergent. If the function $f(t)$ is piecewise continuous on every finite interval in $\langle 0, \infty)$ satisfying the condition $|f(t)| \leq Ke^{st}$ for all $t \in \langle 0, \infty)$, where $K > 0$ and $s \geq 0$, then the integral in (1) exists for all p that fulfil the condition $\mathrm{Re}(p) > s$. Using the Bromwich inversion theorem, the original real function $f(t)$ can be expressed from (1) and for arbitrary real $c > s$ holds

$$f(t) = \mathcal{L}^{-1}\{F(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p) dp. \tag{2}$$

The problem of numerical inverse Laplace transform then lies in the determination of $f(t)$ for discrete values $t > 0$ based on the approximate evaluation of the integral in (2).

Three different types of transforms will be used for testing the precision and the power of selected NILT algorithms in this work. The first one is represented by the transform of sine function, i.e. by $F_1 = 1/(1+p^2)$, which appears in the analytical solution of transient waves in a fixed-free thin rod loaded by a pressure impulse, see [37]. This function also represents a standard test transform which is used in many papers dealing with NILT. It is well known that it is difficult to obtain the sine function from this transform numerically, see [33].

The second transform is taken from [38] and it represents a part of the Laplace transform of the function describing the response of a simply supported viscoelastic orthotropic Timoshenko beam to a pressure impulse of duration t_0. The deflection of such a beam can be expressed as a function of longitudinal coordinates x and time t in the form of the following sine Fourier series [38]

$$v(x,t) = \sum_{n=1}^{\infty} C(n) \sin(\omega_n x) \mathcal{L}^{-1}\left\{ \frac{(1-e^{-pt_0}) H_4(n,p)}{p H_6(n,p)} \right\}, \tag{3}$$

where $C(n)$ characterises the applied load and the other functions in (3) are defined by formulas [38]:

$$H_4(n,p) = {d_0}^2 \left(\rho\, p^2 + {\omega_n}^2 E^*(p) \right) + 12\, \kappa\, G^*(p),$$

$$H_6(n,p) = b_0\, d_0 \left[\left(\rho\, p^2 + \kappa\, {\omega_n}^2 G^*(p) \right) H_4(n,p) - \frac{1}{12} H_2(n,p)^2 \right],$$

$$H_2(n,p) = 12\, \kappa\, \omega_n\, G^*(p), \quad \omega_n = n\pi / l_0,$$

$$E^*(p) = \sum_{k=0}^{N} E_{x_k} - \sum_{k=1}^{N} \frac{{E_{x_k}}^2}{\lambda_{x_k} p + E_{x_k}},$$

$$G^*(p) = \sum_{k=0}^{N} G_{xy_k} - \sum_{k=1}^{N} \frac{{G_{xy_k}}^2}{\eta_{xy_k} p + G_{xy_k}}. \tag{4}$$

The mentioned test transform derived from (3) will be denoted as $F_{2,\,n}$ and will be given by the relation $F_{2,n} = H_4(n,p)/(pH_6(n,p))$. The particular values of parameters present in (4) and used for testing were as follows: the beam length $l_0 = 0.1\,\mathrm{m}$, the height of the beam cross-section $d_0 = 0.005\,\mathrm{m}$, the width of the beam cross-section $b_0 = 0.001\,\mathrm{m}$, the Timoshenko's shear coefficient $\kappa = 0.833$, the material density $\rho = 2250\,\mathrm{kg\,m^{-3}}$, the number of terms in sums $N = 1$, the coefficients of shear and normal viscosity $\eta_{xy_1} = \lambda_{x_1} = 5 \times 10^4\,\mathrm{Pa \cdot s}$, the Young's moduli $E_{x_0} = 35 \times 10^9\,\mathrm{Pa}$, $E_{x_1} = 18.48 \times 10^9\,\mathrm{Pa}$ and the shear moduli $G_{xy_0} = 4 \times 10^9\,\mathrm{Pa}$, $G_{xy_1} = 18.3 \times 10^8\,\mathrm{Pa}$. It is out of scope of this paper to present detailed explanation of each parameter, therefore the interested readers are referred to [38].