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Integration by interpolation and look-up for Galerkin-based isogeometric analysis

Angelos Mantzaflaris[a,*], Bert Jüttler[b]

[a] *Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Linz, Austria*
[b] *Institute of Applied Geometry, Johannes Kepler University, Linz, Austria*

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Highlights

- We propose the new approach of Integration by Interpolation and Lookup (IIL) for matrix assembly in isogeometric analysis.
- The IIL method has a firm theoretical basis: Optimal order of convergence is guaranteed by Strang's Lemma.
- The method circumvents the element-wise assembly that is used in classical FEA.
- The method reduces the overall number of evaluations to one evaluation per physical element, without sacrificing the convergence rate.

Abstract

Even though isogeometric analysis has a clear advantage regarding the number of degrees of freedom needed to achieve a certain level of accuracy, the time needed for matrix assembly (by means of numerical integration of stiffness or mass matrix entries) constitutes a severe bottleneck in the process. In order to overcome this difficulty, we propose the new approach of Integration by Interpolation and Lookup (IIL). Firstly, applying spline interpolation to the common factors in the occurring integrals approximately transforms them into integrals of piecewise polynomial functions, whose integrands are expressed in tensor-product B-spline form. The common factors represent the influence of the geometry mapping (i.e., the NURBS domain parameterization) and the contributions of possibly non-constant material coefficients. Secondly, these integrals are evaluated exactly using pre-computed look-up tables for integrals of tri-products of univariate B-splines and their derivatives. For the model case of elliptic problems, we perform a theoretical analysis to demonstrate that the IIL method maintains the overall approximation order of the Galerkin discretization, provided that the spline interpolation is sufficiently accurate. Moreover, we provide a comparison of the computational complexity of our method with that of a standard Gauss quadrature method. Finally, we present experimental results to illustrate the performance of the IIL method and to support our theoretical results.

Keywords: Isogeometric analysis; Stiffness matrix; Mass matrix; Numerical integration; Quadrature

* Corresponding author. Tel.: +43 73224685279.
E-mail addresses: Angelos.Mantzaflaris@oeaw.ac.at, filiatra@gmail.com (A. Mantzaflaris), Bert.Juettler@jku.at (B. Jüttler).

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1. Introduction

Numerical simulation using Isogeometric Analysis (IGA), which was introduced by Hughes et al. [1,2], relies on NURBS-parameterizations of the computational (physical) domain. When solving (e.g.) a boundary value problem (BVP) via Galerkin projection, the numerical solution is obtained by projecting it into a finite-dimensional discretization space. The basis functions spanning the discretization space are defined using NURBS basis functions and the given domain parameterization. The coefficients of the numerical solution are found by solving a linear system. The latter is formed by mass and/or stiffness matrices and a load vector, possibly combined with advection and other terms. The matrix and right-hand side coefficients of the system have to be computed via numerical integration of piecewise rational multivariate functions. As a major difference to the classical finite element analysis (FEA), exact integration is almost never possible, not even for constant material coefficients, due to the influence of the NURBS domain parameterization (also called the geometry mapping).

Consequently, there are two major sources for numerical error which are equally important: the discretization error and the error introduced by using numerical integration (often called quadrature or cubature in the two- and three-dimensional case, respectively) to evaluate the coefficients and the right-hand side of the linear system. For small- to medium-sized problems, the remaining error source of numerically solving the resulting linear system can safely be neglected.

Similarly to FEA, the standard method for numerically evaluating the integrals in IGA is Gaussian quadrature. Recent research has focused on special quadrature rules that reduce the number of quadrature points and weights used in order to reduce the number of evaluations and to save computing time.

1.1. Related work

Quadrature rules for integrals involving B-splines were already considered more than 30 years ago. In [3] the authors derived rules for the moments of (linear, quadratic and cubic) B-splines, in order to solve a parabolic PDE using Galerkin's method. Since the advent of Isogeometric Analysis, there is an increased interest in this topic, due to its importance for the performance and competitiveness of isogeometric methods.

Optimal quadrature rules for the mass and stiffness of uniform B-spline discretizations are presented in [4]. More precisely, the authors derive rules with the minimum number of nodes that are exact for the product of two B-splines, for degree up to three. The rule is referred to as the half-point rule as the number of nodes (points) plus the number of weights in this minimal rule coincides with the dimension of the spline space of integrands. However, the optimal rule is defined over the whole domain of the B-spline space, and the computation of the nodes and weights leads to a global, non-linear system of equations, which is solved using a Newton iteration. Consequently, the applicability of this approach is limited to low degree and to small number of elements. The authors propose a macro-element strategy to extend the range of applicability to larger number of elements.

The spline space of the product of two uniform B-spline basis functions is further investigated in [5], in order to derive a feasible, computable rule. The basis functions are grouped with respect to the size of their support. This leads to a rule which is defined over one or two elements, and which can be obtained as the solution of a local non-linear system that is independent of the number of elements.

An experimental study of the Gauss rule, and the optimal rule on macro-elements of [4], as well as an extension to degree 4 has been presented in [6]. The authors perform experiments on a Poisson problem over a domain given by the identity mapping, with a unit Jacobian determinant. Their focus is on the degree of exactness of different rules as well as their practical computational cost. Since the parameterization is the identity, the shape functions are simply B-splines, therefore exact evaluation of the stiffness matrix is feasible using Gauss quadrature rules with sufficient degree of exactness.

The use of GPU programming for accelerating the assembly process in isogeometric analysis has been proposed recently by Karatarakis et al. [7,8] based on standard Gauss quadrature rules. In order to exploit the capabilities of modern graphics hardware, the authors present a suitable formulation for the stiffness matrix, which supports the parallelization of the assembly process.

A variety of reduced Bézier element quadrature rules for isogeometric discretizations of low degree have been investigated in [9]. By carefully placing quadrature nodes, preferably on element boundaries, a significant speed-up compared to Gauss quadrature is achieved, due to the reduced number of evaluations per element. As another