Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 284 (2015) 521–540
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Variationally consistent domain integration for isogeometric analysis

M. Hillman, J.S. Chen*, Y. Bazilevs

Department of Structural Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0085, United States

Available online 8 November 2014

Abstract

Spline-type approximations for solving partial differential equations are the basis of isogeometric analysis. While the common approach of using integration cells defined by single knot spans using standard (e.g., Gaussian) quadrature rules is sufficient for accuracy, more efficient domain integration is still in high demand. The recently introduced concept of variational consistency provides a guideline for constructing accurate and convergent methods requiring fewer quadrature points than standard integration techniques. In this work, variationally consistent domain integration is proposed for isogeometric analysis. Test function gradients are constructed to meet the consistency conditions, which only requires solving small linear systems of equations. The proposed approach allows for significant reduction in the number of quadrature points employed while maintaining the stability, accuracy, and optimal convergence properties of higher-order quadrature rules. Several numerical examples are provided to illustrate the performance of the proposed domain integration technique.

Keywords: Isogeometric analysis; Domain integration; Quadrature; Integration constraint; Variationally consistent integration

1. Introduction

Isogeometric analysis (IGA) [1,2] provides a way to link Computer Aided Design (CAD) descriptions of designs directly to analysis, circumventing the lengthy and thus expensive process of producing a suitable discretization, as well as bypassing interaction with CAD descriptions for refinement. It also offers several advantages over traditional finite element analysis such as exact descriptions of geometry, more accuracy per degree of freedom in smooth problems [3], and more favorable transient properties [4], among others. However as with any method cast in the Galerkin framework, numerical integration invariably must be considered since high order quadrature can render numerical methods impractical for analysis.

Spline-, and, in particular, Non-Uniform Rational B-Spline (NURBS)-type approximations for solving partial differential equations (PDEs) are the basis of IGA. B-splines, which form the basis of NURBS, are piece-wise polynomial functions. However, the projected geometry yields approximations that are piece-wise rational and are thus more difficult to integrate than polynomials. The parametric description itself can also become an issue when the mapping from the parametric to spatial domain is not affine with large variations in the Jacobian. The widely adopted approach for domain integration in IGA has been to integrate over cells defined by non-zero knot spans or "elements" using Gaussian quadrature [1], with rules sufficient for exact integration of B-splines with affine mapping.

* Corresponding author.
E-mail address: js-chen@ucsd.edu (J.S. Chen).

http://dx.doi.org/10.1016/j.cma.2014.10.004
0045-7825/

However, this procedure is computationally expensive and has been shown to be suboptimal due to the smoothness of the approximation across cell boundaries [5]. While these cell-by-cell integration rules provide accurate domain integration, quadrature rules that provide the same level of accuracy with fewer quadrature points are in demand [5,6].

One approach that has been taken in IGA to alleviate the issue of higher-order quadrature is to use “macro” integration cells, composed of several cells in each parametric direction, with quadrature rules designed to take into account basis-function continuity across constituent cell boundaries [5]. This resulted in a reduction of the number of quadrature points over conventional Gaussian quadrature, and led to the rules that are “optimal” in the sense they can exactly integrate, with the minimum number of points, the integrands they were designed for, namely, 1D B-splines in the parametric domain. Another approach has been to obtain rules using the translation invariant property of B-splines, with only small non-linear systems of equations to solve in order to obtain quadrature point locations and weights [6]. This results in “nearly optimal” quadrature rules, however, these are only applicable to uniform, structured-mesh configurations. A more unified and less restrictive approach is thus desirable, particularly since unstructured-mesh approximations such as T-splines [7], PHT-splines [8], and locally-refined splines [9] are being rapidly developed and adopted for IGA.

Alternative approaches have been taken outside of IGA in order to alleviate quadrature issues. In particular, in the meshfree method, inefficient domain integration can render the method ineffective, and novel domain integration techniques have been developed to overcome this issue. Rather than redesigning quadrature rules for a given approximation space, test and trial functions are constructed so that accuracy and convergence are achieved with lower order quadrature than would otherwise be required. The approach that is becoming increasingly developed is to impose exactness on the Galerkin method with quadrature for the order of approximation space chosen (cf. [10–12]). The basic idea is that assuming completeness of the trial functions, any solution error when solving a boundary value problem with a solution the same as the completeness order is purely due to numerical integration. Setting the residual of the resulting Galerkin equation as zero results in the so-called integration constraint, and satisfaction of the constraint (Galerkin exactness) in addition to the chosen completeness is taken as a criterion to design the test and trial functions. The earliest example of this technique is the stabilized conforming nodal integration (SCNI) method [10], which has proved to be extremely effective at solving a variety of problems [10,13–16], and has also been applied to other methods such as the natural element method [17]. In this method, nodal integration is employed, and gradients are constructed at the nodes using strain smoothing in order to meet the first order integration constraint for first order Galerkin exactness. More recently, extensions of the strain smoothing technique have been proposed in [11] which meet higher-order constraints.

The above methods fall under the framework of the variational consistency condition proposed in [12]. The condition precisely describes what is necessary in order to obtain nth-order exactness in the Galerkin solution, and is thus an extension of the work in [10]. The work in [12] shows that the quadrature treatment of the integration-by-parts performed starting from the weighted residual of a PDE induces error in the discrete solution. This fact, in turn, may be used as a guideline in constructing quadrature rules (or approximation functions) for a given PDE with far fewer quadrature points than standard techniques and without sacrificing stability and optimal convergence. The authors proposed to construct test functions that meet the integration constraint while keeping the trial functions unmodified to ensure completeness. In this work, variationally consistent domain integration is proposed for IGA. Quadrature cells are defined by non-zero knot spans, and test functions are constructed to meet the variational consistency condition. The method is shown to produce the accuracy of the Galerkin technique with higher-order quadrature while using far fewer integration points. In some cases *one quadrature point per basis function* (for a scalar problem) is employed without degradation of convergence, making the method comparable in computational cost to a collocation technique [18–20].

The remainder of this paper is as follows. Section 2 gives a brief overview of NURBS-based IGA, with a discussion of the properties of the approximation functions relevant to domain integration. The concept of variational consistency is then introduced in Section 3, with emphasis placed on its application to IGA. Numerical examples are then given in Section 4. Concluding remarks are then given in Section 5.

2. NURBS-based IGA

In this section, the NURBS-based isogeometric method is briefly reviewed. Properties of the basis functions relevant to domain integration are also discussed.