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Cell-based maximum-entropy approximants

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Highlights

- Smooth, nonnegative maximum-entropy approximants are constructed on unstructured Delaunay meshes.
- Relative entropy measure is minimized, subject to linear reproducing conditions.
- Nodal prior weight functions are formed by taking powers of smooth approximations to the distance function to the boundary of the support.
- Method delivers optimal convergence rates in Sobolev norms for two-dimensional elliptic boundary-value problems.

Abstract

In this paper, we devise cell-based maximum-entropy (max-ent) basis functions that are used in a Galerkin method for the solution of partial differential equations. The motivation behind this work is the construction of smooth approximants with controllable support on unstructured meshes. In the variational scheme to obtain max-ent basis functions, the nodal prior weight function is constructed from an approximate distance function to a polygonal curve in $\mathbb{R}^2$. More precisely, we take powers of the composition of R-functions via Boolean operations. The basis functions so constructed are nonnegative, smooth, linearly complete, and compactly-supported in a neighbor-ring of segments that enclose each node. The smoothness is controlled by two positive integer parameters: the normalization order of the approximation of the distance function and the power to which it is raised. The properties and mathematical foundations of the new compactly-supported approximants are described, and its use to solve two-dimensional elliptic boundary-value problems (Poisson equation and linear elasticity) is demonstrated. The sound accuracy and the optimal rates of convergence of the method in Sobolev norms are established.

Keywords: Delaunay mesh; Relative entropy; Smooth and nonnegative basis functions; Compact-support; R-functions; Approximate distance function

1. Introduction

Classical finite element (FE) approximations are built on basis functions that possess C^0-continuity. The advent of meshfree methods [1] provided C^k ($k \geq 0$) basis functions that have been adopted in Galerkin methods for the solution

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of second- and higher-order partial differential equations (PDEs). Over the past decade, the importance of smooth basis functions that are also nonnegative has come to the forefront in applications that utilize B-splines, nonuniform rational B-splines (NURBS), isogeometric analysis (IGA) [2,3], subdivision surfaces [4,5] and maximum-entropy (max-ent) approximants [6–9]. The emergence of IGA has highlighted that such smooth basis functions are attractive for the numerical solution of PDEs. The strength of IGA is in high-fidelity boundary-representation, whereas its weakness is in realizing bulk discretizations; hence, coupling it with more flexible methods in the bulk has been explored [10,11]. Such coupling has been done with local max-ent (LME) [6] and with reproducing kernel particle method [12] that use moving least squares (MLS) approximants [13].

A common drawback of meshfree methods, which is particularly acute for large-scale three-dimensional problems, is the very dense nodal-connectivity structure. This leads to many nonzero entries in the stiffness matrix, which in turn places greater demands on memory usage and increases the CPU-times for the numerical simulations. Hence, to realize an efficient and viable Galerkin method on very large nodal sets, smooth basis functions that possess small (minimal) support are desirable. To this end, a method that leverages the complementary strengths of finite elements and spline-based techniques can provide significant advantages vis-à-vis the current state-of-the-art. In this paper, on adopting R-functions [14–16] within the relative entropy variational formulation, we build smooth local approximants that are linearly complete on unstructured meshes. As the first contribution of the new approach presented here, we describe the construction based on two-dimensional Delaunay meshes, and present numerical results for two-dimensional boundary-value problems on such meshes.

In the finite element literature, C^1 shape functions with different number of degrees of freedom (DOFs) for the triangle have been conceived: Argyris triangle (21 DOFs), Bell triangle (18 DOFs) and the composite Hsieh–Clough–Tocher (HCT) triangle (12 DOFs) are well-known [17]. The shape functions in the Argyris triangle span the space of complete fifth-order bivariate polynomials, whereas the Bell triangle is a reduced Argyris triangle. In both constructions, the function value, and its first- and second-order partial derivatives are the nodal unknowns. The HCT triangle is partitioned into three subtriangles and the approximation is a cubic polynomial in each subtriangle. Papanicolopulos and Zervos [18] have presented a systematic framework to derive C^1 shape functions on a triangle. It is noted that compared to linear finite elements on triangular meshes, use of C^1 formulations on such meshes require many more degrees of freedom. Meshfree methods have been tailored for unstructured meshes: Liu et al. [19] proposed the reproducing kernel element method, whereas Duarte et al. [20] constructed arbitrarily smooth generalized finite element approximations. In Ref. [20], a Shepard partition-of-unity function for polygonal domains is constructed using R-functions, which is then multiplied by a linear combination of monomials to form the discrete approximation. In all these previous developments that use finite elements and meshfree approximants, the basis functions that accrue are wiggly and change sign in general.

A simple illustration to demonstrate the effects of varying the support of the nodal basis functions follows. Let us denote the number of rings that surrounds a node by N_R: the one-ring ($N_R = 1$) for a node is the support of standard Delaunay interpolants (piecewise linear finite elements basis functions); the two-ring ($N_R = 2$) is that of subdivision basis functions [4]. In Fig. 1, the nodal basis functions that contribute at a sample point (marked by a $\times$) are shown for LME and for the cell-based max-ent (CME) approach that we propose. LME approximants have a Gaussian decay that is modulated by a nondimensional parameter γ, which controls the aspect ratio and effective support of the resulting basis functions as $h\sqrt{-\log(\epsilon_0)/\gamma}$, where h is the nodal spacing and ϵ_0 is a cutoff-tolerance below which the basis functions are considered to be zero. From Fig. 1, we observe that the nodal neighbors of a sample point for LME approximants with $\gamma = 0.8$ and $\epsilon_0 = 10^{-6}$ corresponds to about the four-ring ($N_R = 4$). The neighbors for $N_R = 2$ and $N_R = 3$ are also depicted, which we use in Section 3 for the CME approximants.

A dense connectivity structure leads to many entries in the system matrix. The sparsity of the system matrix is quantified by the total number of nonzeros (NNZ) of the matrix. We consider the NNZ generated by a set of N points in 2D (3D) that are uniformly distributed in a square (cube), with one degree of freedom per node. In Fig. 2, we show the ratio between the NNZ for max-ent approximants and that of piecewise linear finite element basis functions ($N_R = 1$) in 2D (triangles) and 3D (tetrahedra). A dramatic rise in NNZ with increasing N_R is observed, which is especially pronounced in three dimensions. In 3D, in comparison to the LME approximant ($\gamma = 0.8, \epsilon_0 = 10^{-6}$), the CME ($N_R = 2$) approximants leads to an order of magnitude fewer nonzeros in the system matrix. If cell-based approximants with tighter support can deliver comparable accuracy to standard meshfree basis functions at a significantly less computational cost, then these new basis functions are an attractive choice in Galerkin methods. Furthermore, due to the element-based support, the CME approach can be more easily integrated within existing finite element codes.