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Isogeometric Analysis for second order Partial Differential Equations on surfaces

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Abstract

We consider the numerical solution of second order Partial Differential Equations (PDEs) on lower dimensional manifolds, specifically on surfaces in three dimensional spaces. For the spatial approximation, we consider Isogeometric Analysis which facilitates the encapsulation of the exact geometrical description of the manifold in the analysis when this is represented by B-splines or NURBS. Our analysis addresses linear, nonlinear, time dependent, and eigenvalues problems involving the Laplace–Beltrami operator on surfaces. Moreover, we propose a priori error estimates under h-refinement in the general case of second order PDEs on the lower dimensional manifolds. We highlight the accuracy and efficiency of Isogeometric Analysis with respect to the exactness of the geometrical representations of the surfaces.

Keywords: Second order Partial Differential Equations; Manifolds; Surfaces; Laplace–Beltrami operator; Isogeometric Analysis; A priori error estimation

1. Introduction

In several instances, Partial Differential Equations (PDEs) are set up on lower dimensional manifolds with respect to the hosting physical space, namely on surfaces in three dimensions or curves in two or three-dimensions [1]. Applications include problems in Fluid Dynamics, Biology, Electromagnetism, and image processing as reported for example in [2–6]. In addition, PDEs on lower dimensional manifolds could be obtained as reduced mathematical formulations of PDEs defined in thin geometries, e.g. for plates and shells structures [7].

The numerical approximation of these PDEs generally requires the generation of an approximated geometry compatible with the analysis, as it is the case for the Finite Element method (see e.g. [8–10]). In particular, the approximation of the curvature of surfaces may significantly affect the total error associated to the numerical approximation.

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Typically, schemes based on the Finite Element method have been used for the approximation of PDEs on surfaces with particular emphasis in controlling and limiting the propagation of the errors associated to the discrete geometrical representation. With this aim, surface Finite Element methods [11,12] and geometrically consistent Finite Element mesh adaptations [13,14] have been considered. As alternatives, approaches based on the implicit or immersed surfaces have been proposed, namely based on level set formulations [4,15] or diffuse interfaces strategies [5]. Still, for a broad range of geometries (surfaces) of practical interest, the above mentioned approaches are not error free in the geometrical representation.

As alternative to these approaches, in this paper we propose numerical approximation of PDEs on lower dimensional manifolds by means of Isogeometric Analysis. Our approach is motivated by the fact that a broad range of geometries of practical interest are exactly represented by B-splines or NURBS [16].

Isogeometric Analysis is an approximation method for PDEs based on the isoparametric concept for which the same basis functions used for the geometrical representation are then also used for the numerical approximations of the PDEs [17,18]. Typically, B-splines or NURBS geometrical representations are considered for Isogeometric Analysis, even if, more recently, T-splines [19] have been successfully utilized. Since NURBS are the golden standard in Computer Aided Design (CAD) technology, the use of Isogeometric Analysis facilitates the encapsulation of the exact geometrical representation in the analysis and simplifies the establishment of direct communications between design and numerical approximation of the PDEs. Moreover, NURBS-based Isogeometric Analysis possesses several advantages besides the geometrical considerations, especially in terms of smoothness of the basis functions and accuracy properties [20–22]. Nowadays, Isogeometric Analysis have been successfully used in a broad range of applications in computational mechanics and optimization, see e.g. [17,23–26]. In particular, Isogeometric Analysis have been considered for solving shell problems, as e.g. in [27], and, more recently, Isogeometric Analysis in the framework of the Boundary Element method [28] has been used to take advantage of the exact geometrical representation of surfaces [29].

In this work we provide for the first time a general formulation of the numerical approximation of second order PDEs defined on lower dimensional manifolds described by NURBS, specifically surfaces, by means of Isogeometric Analysis. We discuss the representation of the manifolds in a general framework by means of geometrical mappings from the parameter space to the physical domain; consequently, in view of the use of Isogeometric Analysis based on the Galerkin method [10], we recast the weak forms of the problems and the spatial differential operators in the parameter space. We provide a priori error estimates under h-refinement for the numerical approximation by means of Isogeometric Analysis, thus extending the results of [20,21] to the case of the second order PDEs on lower dimensional manifolds; with this aim, an interpolation error estimate for the NURBS space on the manifold is proposed. We show the accuracy and efficiency of the method by solving several PDEs endowed with the Laplace–Beltrami spatial operator on surfaces. In particular, as few remarkable instances, we address the numerical solution of the Laplace–Beltrami problem, the eigenvalue problem, a time dependent linear advection–diffusion equation, and the Cahn–Allen phase transition equation [30,31]. For both the Laplace–Beltrami and the eigenvalue problems we compare the convergence rates of the errors obtained by means of Isogeometric Analysis with those expected from the a priori error estimates and we highlight the advantages of exactly representing the geometries at the coarsest level of discretization. In this respect, we also compare the numerical results obtained by means of Isogeometric Analysis with those obtained by the standard isoparametric Finite Element method [9].

This work is organized as follows. In Section 2 we discuss the representation of lower dimensional manifolds by NURBS and the role of the parametrization in the definition of geometrical mappings. In Section 3 we consider the PDEs on the manifolds for problems involving the second order Laplace–Beltrami spatial operator. In Section 4 we discuss the numerical approximation schemes, specifically Isogeometric Analysis for the spatial approximation; for the time dependent problems, the generalized-α method [32] is considered and a SUPG stabilization scheme [33] is presented because of its suitability to treat advection dominated problems. In Section 5 we provide the interpolation error and a priori error estimates for h-refined NURBS "meshes". In Section 6 we report and discuss the numerical results for PDEs on surfaces. Final considerations are reported in the Conclusions.

2. Manifolds represented by NURBS

In this section we introduce in an abstract setting lower dimensional manifolds in the physical space, e.g. curves and surfaces, represented by suitable geometrical mappings. We recall the definition of generic functions and their derivatives on the manifold and we express them in terms of the parametric coordinates upon which the geometrical