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An isogeometric collocation method using superconvergent points

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Highlights

- We propose an isogeometric collocation method with improved approximation properties.
- The locations of the collocation points are derived from the superconvergence theory.
- The proposed method achieves optimal convergence rates in the first derivative norms.
- We include a detailed comparison with standard collocation and the Galerkin method.

Abstract

We develop an IGA collocation method modified by collocating at points other than the standard Greville abscissae. The method is related to orthogonal collocation used for solving differential equations and to the superconvergence theory, therefore we refer to this method as "super-collocation" (IGA-SC). By carefully choosing the collocation points, it can be seen that the IGA-SC converges in the first derivative (energy) norms at rates similar to that of the Galerkin solution. This is different from the collocation at Greville abscissae (IGA-C), where the convergence in energy norm for odd polynomial degrees is typically suboptimal. The method is tested on 1D, 2D and 3D numerical examples, in which it is compared to IGA-C and Galerkin's method (IGA-G). The comparison includes a detailed cost vs. accuracy analysis, which shows an improved efficiency of the proposed method in particular for odd polynomial degrees.

Keywords: IGA collocation; Orthogonal collocation; Galerkin method; Greville abscissae; Superconvergence; Least-squares

1. Introduction

Isogeometric analysis (IGA) is a numerical method introduced by Hughes et al. [1,2]. The initial motivation of IGA is to bridge computer aided designed (CAD) and finite element analysis (FEA), through the use of NURBS basis functions for both geometric design and the PDEs analysis. It has been shown to be more accurate than p-FEM on a degree of freedom basis [3–5] while allowing the possibility of smoother (up to C^{p-1}) approximations. In addition, it is possible to obtain more accurate, even exact, geometry representations on very coarse meshes. The method has

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been successfully used in a variety of applications such as plate and shell problems [6–8], fluid mechanics [9–11] and electromagnetism [12].

Most of the commonly used PDE solvers are based on the so-called weak (or variational form) of the PDE. First, the model problem to be solved is described by PDEs which make up the so-called the strong formulation. Then, the spaces corresponding with the trial and the test basis are defined. Next, using the Gauss divergence theorem and integration by parts techniques, the strong formulation is transformed to the weak formulation, which requires less regularity. In the next step, the integrals are evaluated (usually using a numerical quadrature method) and a linear system is assembled. Finally, the linear system is solved to obtain the coefficients corresponding to the basis functions in the approximate solution. While each method has its own particularities, one common feature is that the accuracy of the solutions depends on the quality of the numerical integration. Gaussian quadrature rules are most commonly used and quite efficient, however, as the polynomial degree of the approximation increases, more Gauss points are needed to accurately evaluate the resulting integrals. For the IGA method in particular, Gauss quadrature does not fully take into account the higher continuity of the basis functions which has led to the development of reduced-quadrature methods [13] and other customized quadrature rules [14–18].

A more promising approach, which eliminates integration, is to work directly with the strong form of the PDE. The IGA collocation method [19–23] has been developed since 2010 and, to a large extent, it is combining the accuracy and the smoothness advantages of the IGA method with the computational efficiency of the collocation method. Since there are no volume integrals in the IGA collocation, the method is considerably cheaper from a computational point of view. It is also relatively easy to implement, since it only requires point evaluation of the shape functions and the right-hand side data at the chosen collocation points. The boundary conditions are imposed as additional constraints in the linear system, which is typically non-symmetric even for self-adjoint problem but more sparse compared to the Galerkin method.

Until now, the mathematical theory of collocation has not been very well developed. It has been shown only in 1-dimension that the isogeometric collocation method has optimal convergence of $O(h^{p-1})$ for the 2nd derivative norms, where p is the polynomial degree and h is the maximum element length. However, in the existing numerical studies [19,22], it has been observed that the convergence rate for the first derivative norms when p is odd is also of the order $O(h^{p-1})$, which is suboptimal in comparison to the Galerkin method.

In the present work, we improve the accuracy and the convergence rate of the existing IGA collocation by selecting the collocation point at the zeros of a polynomial defined on a reference interval and scaled to each knot-span. This method is similar to the orthogonal collocation method for B-splines developed by [24,25], where the collocation points are chosen at the Gauss points (or the roots of the Legendre polynomials). However, this requires that each knot is repeated $p-1$ times which means that more degrees of freedom are required and the approximation scheme is only C^1 continuous. To preserve the smoothness of the computed solution, we choose instead the collocation points at the so-called "superconvergent points" for the Galerkin method. This allows an approximation that is close in accuracy to the Galerkin approximation, but at the cost of extra basis function evaluations compared to the standard collocation. An efficiency analysis indicates the added cost of the method (in terms of computational time) is offset by the increased accuracy for odd polynomial degrees. Therefore, this method can be used to complement standard collocation, which has optimal convergence only for even degrees of the polynomial basis.

This paper is organized as follows: In Section 2, we briefly view IGA-G and IGA-C methods, and then present our IGA-SC method. The proposed method is used to solve several problems with known analytical solutions in Section 3 and the performance of the collocation and Galerkin methods is compared. In Section 4, we include a discussion of the algorithmic efficiency of the proposed method. The paper ends with a brief summary in Section 5.

2. IGA collocation methods

In this section, we present in some detail the main ideas of IGA collocation. In the first two subsections, the NURBS basis functions and some basic principles for IGA collocation are introduced. In the remaining subsection, a more detailed mathematical discussion for the new IGA collocation method is presented.

2.1. NURBS

Non-uniform rational B-splines (NURBS) are the basis functions widely used in Computer Assisted Design and also used for analysis in IGA [1]. They are weighted rational B-spline functions. Let $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$ be a