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Robust and optimal multi-iterative techniques for IgA collocation linear systems

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Highlights

- We consider fast solvers for large IgA collocation linear systems.
- We design an optimal and totally robust multi-iterative method.
- The practical behavior is carefully studied through the notion of a symbol.

Abstract

We consider fast solvers for the large linear systems coming from the Isogeometric Analysis (IgA) collocation approximation based on B-splines of full elliptic d-dimensional Partial Differential Equations (PDEs). We are interested in designing iterative algorithms which are optimal and robust. The former property implies that the computational cost is linear with respect to the number of degrees of freedom (i.e. the matrix size). The latter property means that the convergence rate is completely independent of (or only mildly dependent on) all the relevant parameters: in our setting, we can mention the coefficients of the PDE, the dimensionality d, the geometric map $\mathbf{G}$ describing the physical domain, the matrix size (related to the fineness parameters), and the spline degrees (associated with the IgA approximation order). Our approach is based on the spectral symbol, which describes the global eigenvalue behavior of the IgA collocation matrices. It is precisely the spectral information contained in the symbol that is exploited for the design of an optimal and robust multi-iterative solver of multigrid type. Several numerical experiments are presented and discussed in view of recent theoretical findings concerning the symbol and its properties.

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Keywords: Isogeometric analysis; B-splines; Collocation method; Toeplitz matrices; Symbol; P-GMRES, multigrid, and multi-iterative methods

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1. Introduction

We consider the Isogeometric Analysis (IgA) collocation approximation based on B-splines of a full elliptic Partial Differential Equation (PDE) with non-constant coefficients and homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\alpha} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$, $K : \overline{\Omega} \to \mathbb{R}^{d\times d}$ is a Symmetric Positive Definite (SPD) matrix of functions belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, $\boldsymbol{\alpha} : \overline{\Omega} \to \mathbb{R}^d$ is a vector of functions in $C(\overline{\Omega})$, $\gamma, \mathrm{f} \in C(\overline{\Omega})$ and $\gamma \geq 0$.

IgA is a well-established paradigm for the analysis of problems governed by PDEs [1]. Its goal is to improve the connection between numerical simulation and Computer Aided Design (CAD) systems. The main idea in IgA is to use directly the geometry provided by CAD systems and to approximate the unknown solutions of differential equations by the same type of functions. Tensor-product B-splines and their rational extension, the so-called NURBS, are the dominant technology in CAD systems used in engineering, and thus also in IgA. The Galerkin formulation has been intensively employed in this context. However, the efficiency of the Galerkin method deeply depends on the numerical quadrature rules required in the assembly of the corresponding linear systems. In contrast with the finite element context, where elementwise Gauss quadrature is known to be optimal, it is not yet completely understood how to construct efficient IgA oriented quadrature rules; see e.g. [2]. The quadrature issue motivated the use of collocation methods in IgA; see [3] and Section 2.1 for a brief description. The extremely easy and fast construction of their linear systems makes the isogeometric collocation methods very attractive and competitive, mainly for large degrees [4].

In this paper we are interested in the design of fast iterative solvers for the linear systems coming from the isogeometric collocation approximation of (1.1). The intrinsic lack of symmetry of collocation linear systems constitutes an additional difficulty with respect to the Galerkin context. Our approach is based on the spectral information of the corresponding matrices and, in fact, exploits recent results on the asymptotic spectral distribution given in [5]. In several contexts, such information turned out to be useful in the convergence analysis of (preconditioned) Krylov methods (see [6,7] and references therein) and in the design of effective preconditioners and multigrid/multi-iterative solvers (see [8–10]). In [5], the problem (1.1) has been addressed in its full generality and it has been proved that:

1. a spectral distribution for the B-spline isogeometric collocation matrices exists and is compactly described by a symbol f;
2. the symbol f has a canonical structure incorporating:
   (a) the approximation technique, identified by a finite set of polynomials in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \dots, \theta_d) \in [-\pi, \pi]^d$;
   (b) the geometry, identified by the map $\mathbf{G}$ in the parametric variables $\hat{\mathbf{x}} := (\hat{x}_1, \dots, \hat{x}_d)$, defined on the parametric domain $\widehat{\Omega} := [0, 1]^d$;
   (c) the coefficients of the principal symbol of the PDE in (1.1), namely K, in the physical variables $\mathbf{x} := (x_1, \dots, x_d)$ defined on the physical domain Ω.

Most of the analytic features of the symbol derived in [5] are similar to the ones known in the Finite Difference setting [11], where this kind of analysis was originally performed. In particular, if we denote by $\boldsymbol{p} := (p_1, \dots, p_d)$ the vector such that p_j is the spline degree in the $\hat{x}_j$-direction, then f is a nonnegative trigonometric polynomial in the $\boldsymbol{\theta}$-variables, with degree p_j in the variable θ_j, and with a unique zero at $\boldsymbol{\theta} = \mathbf{0}$ of order two. However, a surprising behavior occurs when one of the $\boldsymbol{p}$-parameters, say p_j, becomes large. In this case, the symbol f possesses infinitely many numerical zeros at the points $\boldsymbol{\theta}$ such that $\theta_j = \pi$, and the convergence to zero is monotonic and exponential with respect to p_j. This implies that small eigenvalues of the IgA collocation matrices appear related to high frequency eigenvectors, and this non-canonical source of ill-conditioning is responsible for a significant slowdown of all standard multigrid and preconditioning techniques. On the other hand, the latter information can be exploited for designing ad hoc algorithms with a convergence speed independent of the fineness parameters and of the approximation degrees $\boldsymbol{p}$. Such an approach has been successfully applied in [9] in the context of Galerkin B-spline IgA, by exploiting the asymptotic spectral analysis carried out in [12] for the simplified case where K is the identity matrix and $\Omega = (0, 1)^d$.

We propose the following two-step strategy for solving the linear systems coming from the B-spline IgA collocation approximation of (1.1):