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On stability, convergence and accuracy of bES-FEM and bFS-FEM for nearly incompressible elasticity

Thanh Hai Ong[a], Claire E. Heaney[b], Chang-Kye Lee[b], G.R. Liu[c], H. Nguyen-Xuan[d,e,*]

[a] Department of Analysis, Faculty of Mathematics Computer Science, University of Science, VNU-HCMC, Nguyen Van Cu Street, District 5, Ho Chi Minh City, Viet Nam
[b] Institute of Mechanics and Advanced Materials, School of Engineering, Cardiff University, Queen's Buildings, The Parade, Cardiff CF243AA, UK
[c] School of Aerospace Systems, University of Cincinnati, 2851 Woodside Dr, Cincinnati, OH 45221, USA
[d] Department of Computational Engineering, Vietnamese-German University, Binh Duong New City, Viet Nam
[e] Department of Architectural Engineering, Sejong University, 98 Kunja Dong, Kwangjin Ku, Seoul 143-747, South Korea

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Abstract

We present in this paper a rigorous theoretical framework to show stability, convergence and accuracy of improved edge-based and face-based smoothed finite element methods (bES-FEM and bFS-FEM) for nearly-incompressible elasticity problems. The crucial idea is that the space of piecewise linear polynomials used for the displacements is enriched with bubble functions on each element, while the pressure is a piecewise constant function. The meshes of triangular or tetrahedral elements required by these methods can be generated automatically. The enrichment induces a softening in the bilinear form allowing the weakened weak (W^2) procedure to produce a high-quality solution, free from locking and that does not oscillate. We prove theoretically that both methods confirm the uniform inf–sup and convergence conditions. Four numerical examples are given to validate the reliability of the bES-FEM and bFS-FEM.

Keywords: Finite elements; ES-FEM; FS-FEM; Bubble functions; Volumetric locking; Nearly-incompressible elasticity

1. Introduction

Rubber-like materials are able to withstand extremely high strains whilst exhibiting very little or no permanent deformation and consequently are widely used in industry. In addition to elastic properties, the volume of these materials is almost preserved upon loading. Rubber-like materials are said therefore to be nearly incompressible and typically possess bulk moduli that are several orders of magnitude higher than their shear moduli (equivalently, they have a Poisson's ratio close to one half). It is well known that the stress analysis of nearly-incompressible

* Corresponding author at: Department of Computational Engineering, Vietnamese-German University, Binh Duong New City, Viet Nam.
E-mail address: hung.nx@vgu.edu.vn (H. Nguyen-Xuan).

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materials requires special care. Applying low-order finite elements based on quadrilaterals, hexahedra, triangles or tetrahedra, to such problems, results in a severe underprediction of the displacement known as locking. A variety of numerical methods have been proposed to overcome this defect, for example: h-version of finite elements [1,2], B-bar method [3], mixed formulations [4,5], enhanced assumed strain (EAS) modes [6,7], reduced integration stabilization [8] and two-field mixed stress elements [9], a stream function approach [10] and mimetic finite difference method [11] and so on. In addition to these, several publications investigate an average nodal pressure formulation in which a constant pressure field is enforced over a patch of triangles or tetrahedra [12–16]. Despite the many available approaches for solving nearly-incompressible elasticity problems on a triangulation, only a few methods are based on rigorous mathematical analysis. An example of one such method can be found in [15]. Here, the author introduced a discontinuous pressure and used bubble functions in order to enrich the space of piecewise linear polynomials to which the displacements belong. However, the method still has certain drawbacks inherited from FEM such as (1) an overestimation of the stiffness matrix for nearly-incompressible and bending-dominated problems, (2) a poor performance for distorted meshes, (3) a poor accuracy of the stresses. Moreover, we make mention of the very important three-field (Hu–Washizu) methods. In fact many of the two-field methods mentioned in the overview can be derived as special cases of the Hu–Wahsizu formulation, for which a rigorous analysis has been carried out in [17,18].

In this paper we propose two improved methods which use bubble functions as enrichments to the edge-based and face-based smoothed finite element methods (bES-FEM and bFS-FEM). These methods contribute to the further development of advanced numerical tools that can be used for nearly-incompressible elasticity problems, whilst simultaneously building on the advantages of some classical methods as explained below.

Firstly, an improved version of a so-called bES-FEM has the same desirable features as bES-FEM-T3 studied in [19]. Both bES-FEM and bFS-FEM work well for three-dimension problems, where bubble functions are generally defined by the $(d+1)$th-power bubble function and the hat function. Most importantly, both methods are theoretically proven to ensure the uniform inf–sup condition and the convergence. In addition, there is a basic difference between bES-FEM and bES-FEM-T3 as follows: for bES-FEM, the approximate pressure and displacement are directly computed by the mixed approach provided in (16a) and (16b) while for bES-FEM-T3, the approximate pressure is computed as a posteriori of the displacements based on the edge-based smoothing domains.

Secondly, we use mixed methods [4,20] to reformulate the linear elasticity problem as a mixed displacement–pressure problem. Our aim is to attain a good approximation to the pressure solution [1], which we model here as piecewise constant.

Thirdly, the proposed approximation to the displacement solution is a combination of the displacement from - ES-FEM/FS-FEM [21,22] and the displacement from the bubble functions [23,24]. ES-FEM and FS-FEM improved the standard FE strain fields via a strain smoothing technique described in [25]. The methods proposed in this paper build on ES-FEM and FS-FEM, and therefore inherit the positive qualities associated with this smoothing technique, namely (1) its solutions are more accurate than those of linear triangular elements (FEM-T3) and quadrilateral elements (FEM-Q4) using the same sets of nodes [3,26]; (2) ES-FEM and FS-FEM perform well with distorted meshes; (3) their stress solutions, which are very precise, have the convergent property and (4) they can be easily implemented into existing FEM packages without requiring additional degrees of freedom. Clearly this technique of smoothing is a powerful tool and it has already been applied to a wide range of practical mechanics problems, e.g., [27–29]. Nevertheless, if the displacement is approximated only by ES-FEM or FS-FEM *i.e.* without enrichment by bubble functions, these methods violate the inf–sup condition and uniform convergence. Other methods from the SFEM family also fail to satisfy this condition, implying that they also suffer from volumetric locking in the case of nearly-incompressible elasticity [22,19]. To overcome volumetric locking for the SFEM family, only a few approaches have been presented. For example, in [22], the authors suggested a combined FS/NS-FEM model and in [19] the use of bubble functions was proposed. Neither of these approaches are based on a rigorous mathematical analysis.

Finally, the degree of freedom which is associated with the pressure variable can be statically condensed out of the system of equations, in contrast to the method based on the classical MINI element [20], for example, where condensation cannot be applied.

The rest of this paper is organized as follows. In Section 2, we briefly recall the boundary value problem of linear elasticity, the mixed displacement–pressure formulation and its associated weak form. Section 3 describes the enrichment of ES-FEM and FS-FEM by bubble functions. Section 4 presents the mathematical properties of bES-FEM and bFS-FEM, where only small deformations are considered. Displacement, energy and pressure error norms are defined in Section 5 for the precise quantitative examination of various models. Four numerical tests are presented in Section 6