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Spurious transients of projection methods in microflow simulations

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Highlights

- Most popular fractional step methods are analyzed for highly viscous flows.
- Most incremental methods develop a spurious transient, independent of mesh size.
- Non-incremental methods do not develop spurious transients, but require small time steps.
- Time step bounds are proposed as to avoid artifacts produced by spurious transients.

Abstract

The temporal behavior of projection methods for viscous incompressible low-Reynolds-number flows is addressed. The methods considered result from algebraically splitting the linear system corresponding to each time step, in such a way that the computation of velocity is segregated from that of pressure. Each method is characterized by two (possibly equal) approximate inverses (B_1 and B_2) of the momentum-equation velocity matrix, plus a parameter γ which renders the method non-incremental (if $\gamma = 0$) or incremental (if $\gamma = 1$). The classical first-order projection method, together with more sophisticated methods (Perot's second-order method, Yosida method, pseudo-exact factorization method) and their incremental variants are put into the same algebraic form and their accuracy numerically tested. Splitting errors of first, second and third order in the time step size δt are obtained, depending on the method. The methods are then discussed in terms of their ability and efficiency to compute steady states. Non-incremental methods are impractical because extremely small time steps are required for the steady state, which depends on δt, to be reasonably accurate. Incremental methods, on the other hand, either become unstable as δt is increased or develop a remarkable *spurious transient* which may last an extremely long time (much longer than any physical time scale involved). These transients have serious practical consequences on the simulation of steady (or slowly varying), low-inertia flows. From the physical viewpoint, the spurious transients may interfere with true slow processes of the system, such as heat transfer or species transport, without showing any obvious symptoms (wiggly behavior in space or time, for example, do not occur). From the computational viewpoint, the limitation in time step imposed by the spurious transient phenomenon weighs against choosing projection schemes for microflow applications, despite the low cost of each time step.

Keywords: Projection method; Algebraic splitting; Pressure segregation; Navier–Stokes equations; Incompressible flow; Microfluidics

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1. Introduction

All along the history of the numerical simulation of incompressible flows, much attention has been devoted to *pressure-segregation* methods, which are methods that avoid the solution of linear systems in which the velocity unknowns are coupled to the pressure unknowns. Many pressure-segregation methods have been developed since the early work of Harlow and Welch [1]. We consider here a class of pressure-segregation methods known as *projection methods* [2–7], *pressure-correction methods* [8,9] or *fractional step methods* [10,11]. They have in common that the update of velocity unknowns is obtained from the momentum equation, while the update of pressure unknowns results from solving a Poisson-like equation.

Let us recall the incompressible Navier–Stokes equations, which can be written as

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{uu}) - \nu \nabla^2 \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } [0, T] \times \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } [0, T] \times \Omega, \tag{2}$$

where t is time, $\mathbf{u}$ is the velocity vector field, p is the pressure, ν is the kinematic viscosity of the fluid, $\mathbf{f}$ is a body-force, all defined in a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) for $t \in [0, T]$.

Formally, the above equations are equivalent to

$$\partial_t \mathbf{u} = \Pi_Z \left(\mathbf{f} - \nabla \cdot (\mathbf{uu}) + \nu \nabla^2 \mathbf{u} \right) \tag{3}$$

where Π_Z is the projection operator onto the subspace Z of divergence-free vector fields in Ω. Notice that p does not appear in (3). The early works of A. Chorin [3] and R. Temam [4] exploited the projection structure of the problem to define *projection methods* which successfully accomplish pressure segregation. The underlying mathematics is rooted on the Helmholtz–Hodge theorem (see [12]) about the decomposition of a vector field into gradient and solenoidal components, which allows for the computation of the projection operator by solving a *Poisson equation* for the pressure.

Over the years, many authors [5,13–17] studied the effects of different update schemes for the pressure to obtain higher-order (in time) methods. The treatment of boundary conditions also focused research efforts, as they impact on the time accuracy [15,18,19,7,20–22].

An interesting viewpoint of projection methods is that of *approximate factorization*, or *algebraic splitting* [11,23–25]. In this approach, the decoupling is performed on the spatially and temporally discretized equations by replacing the system matrix of the monolithic method (also called all-at-once method) by an approximate factorization of it. This approach leaves the treatment of the boundary conditions implicit in the approximate matrix factors, much simplifying the analysis. Interesting applications of the approximate factorization approach can be found in the studies by Quarteroni et al. [26], Badia and Codina [27], Lee et al. [20], Chang et al. [28], Griffith [29], among others.

From the cited references it becomes clear that most of the existing projection methods are in fact equivalent to some algebraic splitting method. We have thus chosen to focus this article on algebraic splitting methods, considering them as representative of most projection-like methods.

Section 2 of this article contains an overview of several algebraic splitting methods for viscous incompressible flows, similar to that performed in [25]. The methods are described within a uniform algebraic setting and numerically tested in the benchmark problem of decaying vortices in a periodic domain at $Re = 10^{-2}$. A detailed study of the convergence of the time discretization is conducted, which serves both as verification of the implementation and as direct comparison of the different variants in a low-inertia flow. A similar study involving monolithic and segregated methods for incompressible fluid flows was presented by Elman [30], though in that paper the goal was to investigate the performance of preconditioning strategies (see also [31]).

The convenience of segregating the velocity from the pressure unknowns without sacrificing temporal accuracy is quite significant, as in a mesh with N cells in 3D one solves four $N \times N$ matrices instead of one large matrix of dimensions $4N \times 4N$. Unfortunately, there exist serious time-step restrictions on the applicability of projection methods to low-inertia incompressible flows.

Section 3 discusses these restrictions, which arise from either accuracy or stability considerations. In particular, it is shown that incremental projection methods suffer from severe spurious transients in the computation of pressure-driven viscous-dominated flows. These spurious transients may easily last thousands of time steps. They may become larger than the physical time scale of the process being simulated and completely pollute the numerical results.