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Generalized T-splines and VMCR T-meshes

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Abstract

The paper considers the extension of the T-spline approach to the Generalized B-splines (GB-splines), a relevant class of non-polynomial splines. The Generalized T-splines (GT-splines) are based both on the framework of classical polynomial T-splines and on the Trigonometric GT-splines (TGT-splines), a particular case of GT-splines. Our study of GT-splines introduces a class of T-meshes (named VMCR T-meshes) for which both the corresponding GT-splines and the corresponding polynomial T-splines are linearly independent. A practical characterization can be given for a sub-class of VMCR T-meshes, which we refer to as weakly dual-compatible T-meshes, which properly includes the class of dual-compatible (equivalently, analysis-suitable) T-meshes for an arbitrary (polynomial) order.

Keywords: T-spline; T-mesh; GB-spline; Analysis-suitable; Dual-compatible; Linear independence

1. Introduction

In the last years, the introduction of the so-called T-splines and of the spline spaces defined over T-meshes introduced significant advancements for the use of polynomial spline functions in the CAD and CAGD techniques. The main idea of this approach, in the basic case of surface modeling in $\mathbb{R}^3$, is to free the control points of the surface from the constraint to lie, topologically, on a rectangular grid whose edges intersect only at "cross junctions", and allow instead partial lines of control points, which leads to the possibility to have "T-junctions" between the edges of the grid. Such a framework gave some important improvements in CAD and CAGD methods: the possibility to locally refine the surfaces, a considerable reduction of the quantity of control points needed, the ability to easily avoid gaps between surfaces to be joined (see, e.g., [1,2]), just to name a few. All these advantages became even more important in the applications, such as the isogeometric approach for the analysis problems represented by partial differential equations (see, e.g., [3–5]).

The T-spline idea has been applied mainly to polynomial splines, while we know that several types of non-polynomial splines are used for certain applications because of their particular properties. For this reason, recently

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we proposed a generalization of the T-spline approach to the trigonometric GB-splines (see [6]), a particularly relevant class of non-polynomial splines because of their adaptability and their application to the already mentioned isogeometric analysis (see, e.g., [7,8]). Roughly speaking, the GB-splines are a basis of spaces of piecewise functions, locally spanned both by polynomials and by non-polynomial functions, which in the trigonometric case are $\sin(\omega s)$ and $\cos(\omega s)$, with a given frequency ω. Note that these splines can be seen as particular cases of the piecewise Extended Chebyshevian splines (see, e.g., [9–11]). GB-splines have been successfully used to construct tensor-product surfaces (see, e.g., [8] and references therein) with control points on rectangular grids.

In this paper, we will first extend the results in [6] to any type of GB-splines of *arbitrary bi-order* (p, q), so that we can take full advantage of the good features of GB-splines and T-splines. In order to achieve this goal, we will start by presenting the univariate GB-splines and their properties, including a knot insertion formula with necessary conditions, which will be also essential in the study of the linear independence of the GT-spline functions. Then, we will introduce the GT-splines, whose definition (and notations) is based both on the polynomial T-splines (see, e.g., [12–14]) and on the TGT-splines (see [6]). Similarly to the case of TGT-splines [6], we will show that there exists a relation between the GT-splines of bi-order (p, q) and the polynomial T-splines of the same bi-order. The study of their linear independence will lead us to the introduction of the class of VMCR T-meshes (Void Matrix after Column Reduction T-meshes), which guarantee the linear independence of the associated GT-spline and T-spline blending functions of the same bi-order. The basic concept behind VMCR T-meshes involves the idea of column reduction (used in [14]), and implicitly helped to show in [6] that *in the case of bi-order* (4, 4) the well-known analysis-suitable T-meshes are also VMCR T-meshes. In this paper we provide a simple characterization of a sub-class of VMCR T-meshes, which we refer to as *weakly dual-compatible T-meshes*: we will prove that such class strictly includes the one of dual-compatible/analysis-suitable T-meshes (see, e.g., [13,14]) for any bi-order (p, q). Finally, we will present an explicit example of weakly dual-compatible T-meshes which is not dual-compatible/analysis-suitable.

The paper is organized as follows. In Section 2 we recall the definition and the basic properties of the univariate GB-splines, and we deal with the conditions needed to get a knot insertion formula. In Section 3, after having recalled the definition of T-mesh, we introduce the GT-splines and we give some properties following directly from their definition. In Section 4 we study the linear independence of the GT-spline blending functions and, more importantly, the classes of VMCR T-meshes and of weakly dual-compatible T-meshes. Finally, Section 5 contains some concluding remarks.

2. Univariate generalized B-splines

2.1. Definition and main properties

Let $n, p \in \mathbb{N}$, $p \geq 2$, and let $\mathbf{\Sigma} = \{s_1 \leq \cdots \leq s_{n+p}\}$ be a non-decreasing knot sequence (*knot vector*); we associate to $\mathbf{\Sigma}$ two vectors of functions $\mathbf{\Omega_u} = \{u_1(s), \ldots, u_{n+p-1}(s)\}$ and $\mathbf{\Omega_v} = \{v_1(s), \ldots, v_{n+p-1}(s)\}$, where, for $i = 1, \ldots, n+p-1$, u_i, v_i belong to $C^{p-2}[s_i, s_{i+1}]$ and are such that the space W spanned by the derivatives

$$U_i(s) = \frac{d^{p-2}u_i(s)}{ds^{p-2}}, \qquad V_i(s) = \frac{d^{p-2}v_i(s)}{ds^{p-2}}$$

is a Chebyshev space, that is, any function belonging to it has at most one zero in $[s_i, s_{i+1}]$. We remark that this assumption corresponds to simple conditions in some noteworthy cases (see also [6,15]): if $u_i(s) = \cos \omega_i s$, $v_i(s) = \sin \omega_i s$, it is sufficient to choose ω_i such that $0 < \omega_i < \pi/(s_{i+1} - s_i)$, and if $u_i(s) = \cosh \omega_i s$, $v_i(s) = \sinh \omega_i s$, we have no restrictions on the choice of ω_i.

Let, for $i = 1, \ldots, n+p$, m_i be the multiplicity of s_i in $\mathbf{\Sigma}$, that is, the cardinality of the set

$$\{k : 1 \leq k \leq n+p,\ s_k = s_i\}.$$

Note that $m_i = m_j$ if $s_i = s_j$. We assume that $1 \leq m_i \leq p$, for $i = 1, \ldots, n+p$. We consider the *generalized spline space* spanned, in each interval $[s_i, s_{i+1}]$, by $\{u_i(s), v_i(s), 1, s, \ldots, s^{p-3}\}$ for $p \geq 3$ and by $\{u_i(s), v_i(s)\}$ for $p = 2$. For this space we can define a basis of compactly-supported splines, which are called *Generalized B-splines* (GB-splines).

The definition of such basis is usually given in a recursive fashion, which we briefly recall (see also [7,8]). Since we required that the space spanned by U_i and V_i, denoted by $W = \langle U_i, V_i \rangle$, is a Chebyshev space, it is not restrictive