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A continuation problem for computing solutions of discretised evolution problems with application to plane quasi-static contact problems with friction

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Abstract

A continuation problem for finding successive solutions of discretised abstract first-order evolution problems is proposed and a general piecewise C^1 continuation problem is studied. A condition ensuring local existence and uniqueness of its solution curves is given. An analogy of the first-order system of smooth problems is derived and results of existence and uniqueness of its solutions are stated. Possibility of continuation of a solution curve along directions solving the first-order system is discussed. A technique for numerical continuation of the solution curves is developed. Furthermore, an application of the abstract continuation problem is presented for plane quasi-static contact problems with friction. Various formulations of the first-order system are derived for this case so that the analysis from the abstract frame can be developed and supplemented. Finally, the proposed numerical continuation is tested on model examples.

Keywords: Continuation; Piecewise-smooth system; First-order system; Predictor–corrector; Contact problem; Coulomb friction

1. Introduction

When time-stepping schemes are used to solve quasi-static problems in solid mechanics numerically, one can encounter situations where usual solvers (for instance, the Newton method with the initial approximation chosen to be a solution from the previous time step) fail to compute any solution. Typically, this can happen when a snap-through instability is present and even a small change in loading leads to a dramatic change of the solution. This has lead us to construct a suitable continuation problem for dealing with such situations. Although our motivation originated

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from solving quasi-static problems with the particularity that the time derivative appears only in a nonlinear term, our approach can be applied to first-order evolution problems generally. The idea how to do it is explained in an abstract frame in the beginning of Section 2.

Whereas continuation is well-established for problems involving a continuously differentiable map (see [1], for example), a little work has been done for problems with general non-smooth functions and it is oriented mainly to homotopy methods [2,3]. That is why the next ambition of this paper is to give a rigorous analysis of a general continuation problem. In particular, a problem involving an arbitrary piecewise C^1 (PC^1) function is considered in Section 2.1, and a result guaranteeing local existence and uniqueness of its solution curves is stated. Furthermore, an analogy of the first-order system for smooth problems is introduced, which gives a possibility of studying tangent behaviour of solution curves near a given solution point.

The reason why we confine ourselves to the framework of PC^1-functions is that it seems to be well-suited for plane contact problems, for which a particular continuation problem is proposed in Section 3. Contact problems lead to functions that are not Gâteaux-differentiable in general. Nevertheless, let us note that other problems from engineering or economics are covered by the framework, as well [4].

After laying theoretical foundation, we describe a method of numerical continuation for tracing PC^1 solution curves in Section 2.2. Our approach is close to the continuation from [3] for normal maps with polyhedral convex sets, to the ones from [5,6] for frictionless contact problems or to the ones from [7–9] for plane contact problems with Coulomb friction. The main contrast to all those papers is that the present algorithm does not obey precise expressions of sub-domains of smooth behaviour of the PC^1-function involved, which require quite detailed specification of the function during implementation.

The strategy proposed here is based on the predictor–corrector method for smooth functions sketched in [10], which resembles an arc-length continuation and is capable of traversing smooth folds on its own, without adding any special routines. Having inherited this property, our continuation includes also a special technique for treating non-smooth points on solution curves, which may be even fold points at the same time. Let us point out that we use no smoothing unlike [11,12] to avoid the danger of modification of the solution structure of the original non-smooth problem.

As indicated, Section 3 deals with a particular continuation problem for discretised quasi-static plane contact problems with Coulomb friction in large deformations. Firstly, we formulate the corresponding continuous problem and discretise it. The spatial discretisation is done by a mixed finite-element method while time derivatives are approximated by backward differences. Then we propose a continuation problem for this case and making use of ideas from [13], we reformulate it so that it fits our general framework.

Section 3.1 employs the specific structure of the problem and establishes more precise analysis of its first-order system, extending the studies [5,6,14] for frictionless problems. In particular, we formulate the first-order system in such a way that it is close to a rate problem of a quasi-static contact problem from the mathematical point of view. Making use of this similarity, we adapt the analysis of the rate problem from [15] to our first-order system. Moreover, we investigate the abstract result of continuation of solution curves in directions solving the first-order system.

Finally, numerical experiments with the continuation method from Section 2.2 are presented in Section 3.2. A similar algorithm has already been tested on static contact problems in [7,8], but only on finite-element models with very small number of degrees of freedom. Here we show results for more realistic models.

1.1. Notation and preliminaries

The following notation is employed throughout the paper: For a vector $\boldsymbol{x} \in \mathbb{R}^N$, a matrix $\boldsymbol{A} \in \mathbb{R}^{M \times N}$ and index sets $I \subset \{1, \ldots, M\}$ and $J \subset \{1, \ldots, N\}$, x_i stands for the ith component of $\boldsymbol{x}$, $\boldsymbol{x}_J$ is the sub-vector of $\boldsymbol{x}$ composed from the components x_i, $i \in J$, and $\boldsymbol{A}_{I,J}$ is the sub-matrix of $\boldsymbol{A}$ with rows and columns specified by I and J, respectively. Furthermore, $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{y}$ is a scalar product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\|\boldsymbol{x}\|$ denotes the Euclidean norm of $\boldsymbol{x}$ and $B(\boldsymbol{x}, r)$ stands for a closed ball centred at $\boldsymbol{x}$ with radius r.

The gradients of a real-valued function f and a vector-valued function $\boldsymbol{f}$ at a point $\bar{\boldsymbol{x}}$ are denoted by $\nabla f(\bar{\boldsymbol{x}})$ and $\boldsymbol{\nabla f}(\bar{\boldsymbol{x}})$, respectively, the partial gradients with respect to $\boldsymbol{y}$ of f and $\boldsymbol{f}$ at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ are denoted by $\nabla_{\boldsymbol{y}} f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and $\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$, respectively, and $\nabla^2 f(\bar{\boldsymbol{x}})$ is the Hessian of f at $\bar{\boldsymbol{x}}$. If $\boldsymbol{f}$ is a function of a real variable, $\boldsymbol{f}'$ denotes its right-hand derivative for brevity of notation. In the same way as for vectors, the jth component function of a vector-valued function $\boldsymbol{f}$ is systematically denoted by f_j.

For reader's convenience, we recall essentials from theory of PC^1-functions [4,16]: