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Natural vorticity boundary conditions on solid walls

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Highlights

- We devise vorticity boundary conditions on solid walls, which possess physical and geometrical information relevant to near-boundary vortex dynamics.
- The conditions are local and enforced in the Galerkin formulation through a right-hand side functional depending on the pressure.
- Two numerical schemes are suggested that benefit from the new conditions and solve for velocity, vorticity, and pressure in a decoupled time-stepping fashion.

Abstract

We derive a new kind of boundary conditions for the vorticity equation with solid wall boundaries for fluid flow problems. The formulation uses a Dirichlet condition for the normal component of vorticity and Neumann type conditions for the tangential components. In a Galerkin (integral) formulation the tangential condition is natural, i.e., it is enforced by a right-hand side functional and does not impose a boundary constraint on trial and test spaces. The functional involves the pressure variable, and we discuss several velocity–vorticity formulations where the proposed condition is appropriate. Several numerical experiments are given that illustrate the validity of the new approach.

Keywords: Incompressible Navier–Stokes equations; Vorticity boundary conditions; Finite element method; Vorticity–velocity formulation

1. Introduction

Fluid flow vorticity is an important dynamic variable and many phenomena can be described in terms of vorticity more readily than in terms of primitive variables. Vorticity plays a fundamental role in understanding the physics of laminar, transitional and turbulent flows [1–3], in mathematical analysis of fluid equations [4], and in computational fluid dynamics [5].

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The vorticity dynamics of incompressible viscous fluid flows is driven by the system of equations

$$\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{u} = \nabla \times \mathbf{f} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity in a non-inertial reference frame, $\mathbf{w} = \nabla \times \mathbf{u}$ is the flow vorticity, ν is the kinematic viscosity coefficient, and $\mathbf{f}$ is a vector function of body forces per unit mass. To obtain a closed system, one should complement (1) with equations for $\mathbf{u}$ and initial conditions, and if a flow problem is posed in domain with boundaries, then boundary conditions should be prescribed. Commonly, boundary conditions are given in terms of primal variables and stress tensor, rather than in terms of the vorticity. However, for analysis and for numerical methods based on vorticity equations, it is important to endow (1) with boundary conditions on $\mathbf{w}$.

Appropriate vorticity boundary conditions have been a subject of intensive discussion in the literature, especially, in the context of numerical methods for fluid equations. In Section 2, we include a brief review of several main approaches. One should be especially careful with assigning vorticity boundary conditions on solid walls, i.e. those parts of the boundary where a fluid is assumed to have no-slip velocity, since these regions are responsible for vorticity production and give rise to physical and numerical boundary layers. An obvious choice of using the vorticity definition $\mathbf{w} = \nabla \times \mathbf{u}$ for the boundary condition on $\mathbf{w}$ is not always optimal with respect to numerical accuracy [6,7]. This motivated our search for an alternative way of prescribing boundary conditions on $\mathbf{w}$.

The main result of this paper is that on a no-slip boundary, a vorticity boundary condition is derived that depends on the tangential pressure gradient, and can be efficiently implemented as a natural boundary condition in variational methods. Natural boundary conditions are easy to implement numerically in variational methods, since they do not impose boundary constraints on trial and test spaces in a Galerkin method, and are less prone to produce numerical boundary layers. The critical role of tangential pressure gradients for boundary vorticity generation is known in the literature and discussed, e.g., in [8,3], however this relationship has seemingly not been exploited for devising numerically efficient boundary conditions.

The vorticity boundary condition is derived in Section 3, after providing necessary preliminaries in Section 2. In addition to the derivation, we also discuss in Section 3 what new insight the boundary conditions may give in a possible role of pressure and surface curvature in the vorticity production along solid boundaries. The remaining Sections 4 and 5, are dedicated to potential applications of the derived vorticity boundary condition. Section 4 discusses several options to close the system of equations by combining (1) with the vector Poisson equations for velocity, or different formulations of the momentum equations with nonlinear terms driven by the Lamb vector. Section 5 presents results of several numerical experiments which demonstrate the utility and efficiency of the new vorticity boundary conditions for computing incompressible viscous flows. Finally, Section 6 collects a few closing remarks.

2. Problem setup and boundary conditions review

We consider the flow of an incompressible viscous Newtonian fluid in a bounded domain $\Omega \in \mathbb{R}^3$. In primitive (velocity–pressure) variables, the fluid motion is described by the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{2}$$

We distinguish between the upstream (inflow), downstream (outflow) and no-slip parts of the boundary, Γ_{in}, Γ_{out}, and Γ_w ($\partial\Omega = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_w$), to impose different types of boundary conditions on them. On Γ_{in} we assume a prescribed velocity profile $\mathbf{u}_{in}$ and an outflow boundary condition on Γ_{out}, e.g., the vanishing normal component of the stress tensor [9]. On the no-slip boundary Γ_w we have

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_w, \text{ with } \mathbf{g} \cdot \mathbf{n} = 0, \tag{3}$$

where $\mathbf{n}$ is an outward normal vector for Γ_w and $\mathbf{g}(\mathbf{x}, t)$ is a tangential velocity of the solid part of boundary. It is common to have $\mathbf{g} = \mathbf{0}$ for flows past a steady object or channel flow.

For the inflow, one may assume vorticity is known, and set $\mathbf{w} = \nabla \times \mathbf{u}_{in}$ on Γ_{in}. Reasonable conditions for the outflow [10,11] are letting the normal vorticity derivative vanish: $(\nabla \mathbf{w})\mathbf{n} = \mathbf{0}$ on Γ_{out}, other Neumann conditions for