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An adaptive sparse grid method for elliptic PDEs with stochastic coefficients

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Abstract

The stochastic collocation method based on an anisotropic sparse grid is nowadays a significant tool to solve partial differential equations with random input data. This method is based on a level of interpolation and weights of anisotropy. The objective of some adaptive approaches is to select cleverly these parameters, in order to reduce the computational cost.

In this work, we propose such an adaptive approach, based on an approximation of the *inverse* diffusion coefficient. We introduce an error indicator which is an upper bound of the error in the solution and use this indicator as a reliable and cheap tool for choosing the level of interpolation. We also propose a new error estimation in one dimension, for unbounded random variables, and use it to compute suitable weights.

Numerical examples show the efficiency of our methodology, since the cost is considerably reduced, without loss of accuracy.

Keywords: Elliptic PDEs with random input data; Stochastic collocation method; Anisotropic sparse grid; Adaptive method

1. Introduction

The Monte Carlo method [1] is the most standard approach used to compute statistical quantities of interest depending on the solution of partial differential equation with stochastic inputs. It consists of solving M deterministic problems, where M is the number of independent and identically distributed simulations (iid) of the parameters. The main disadvantage of this approach is its slow convergence given by the order $O(\frac{1}{\sqrt{M}})$, hence the method requires in general a large computational effort.

Recently, spectral methods have been developed, such as stochastic finite element and collocation methods. They offer a robust tool for solving problems of stochastic PDE [2–6]. They approach the response of the model as a stochastic function by a polynomial interpolation in the stochastic space, and provide an exponentially convergent

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approximation when the solution of the problem is a smooth function with respect to the random variables. However, stochastic finite element methods require solving a large problem combining physical and probability spaces, whereas collocation methods require solving many deterministic problems in the physical space. These methods suffer from a curse of dimensionality, since the computational cost increases exponentially with the stochastic dimension. This often limits their application to problems with a small stochastic dimension. When the stochastic dimension is moderately large, it is necessary to minimize the computational cost.

Reduced basis methods [7–10] are a first approach for problems with linear random coefficients. A first offline–online step and a posteriori error estimates are used to find the most representative samples of the solution and to build a reduced basis. The physical deterministic problem is projected into this reduced space to get an approximation at a lower cost. The stochastic collocation method can be combined with this approach [8,11].

Sparse and anisotropic polynomial interpolation aims at minimizing the total number of collocation points. The anisotropic method proposed in [12] uses a priori and a posteriori information based on the regularity of the solution, while the adaptive method developed in [13] and applied to the problem of PDEs in [14] defines the level of the approximation by successive enrichments. The approach developed in [15,16] uses sparse wavelet chaos subspaces for linear elliptic PDEs. These approximations are also anisotropic and exploit the decay of the eigenvalues in the expansion of the diffusion coefficient to define an approximation in the chaos polynomial space.

In this work, we focus our attention to collocation with an anisotropic sparse grid. We develop an adaptive approach to estimate the weights and the level of interpolation in an anisotropic sparse grid, in the same spirit as [4,15,12]. In [17], we introduced the Karhunen–Loève expansion of the inverse of the diffusion coefficient, and used the basis of this expansion to compute the average solution of a one dimensional elliptic problem. Motivated by this work, we now use this inverse of the diffusion to define an error indicator. We prove that this indicator satisfies an upper bound of the interpolation error in the solution and we provide estimations of the constant in this bound. This error indicator can be used to select a level of interpolation. Then we prove a new error bound for problems with one-dimensional unbounded random variables. We use our indicator and one-dimensional error bounds to compute the weights of anisotropy. This process requires only a sequence of cheap interpolation problems, without solving expensive deterministic problems in the physical space.

The paper is organized as follows. In the first section, we introduce the mathematical problem setting with a finite noise assumption. In Section 2, we recall the stochastic collocation methods based either on a full tensor product or a sparse grid. In Section 3, we develop our methodology and our theoretical results. Finally, in Section 4, we present numerical examples to illustrate the efficiency of our approach.

2. Problem formulation

Let $(\Omega, \mathcal{F}, d\mathbb{P})$ be a complete probability space, where Ω is the space of elementary event, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of events and $\mathbb{P}$ is the probability measure. Also, we consider a bounded domain $D \subset \mathbb{R}^d$, with a smooth boundary ∂D. The input data k and f are two random fields on $\Omega \times D$. We focus on the following linear elliptic boundary value problem: find a stochastic function, $u : \Omega \times D \longrightarrow \mathbb{R}$ such that the following equation holds (a.s) in Ω:

$$(\mathcal{P}_s) \begin{cases} -\operatorname{div}(k\nabla u) = f & \text{in } \Omega \times D, \\ u\,|_{\partial D} = 0. \end{cases} \tag{1}$$

We make the following assumptions on the random input data k and f to preserve the well-posedness of the problem $(\mathcal{P}_s)$.

Assumption 1.
- **H1**: f belongs to $L^p(\Omega) \otimes L^2(D)$, for all $p \in [1, \infty[$.
- **H2**: There exist k_{min}, k_{max} such that, for each $\omega \in \Omega$ (a.s) we have $0 < k_{min}(\omega) \le k(\omega, .) \le k_{max}(\omega)$ and $k_{min}^{-1} \in L^p(\Omega)$.

Let $H_0^1(D)$ be the subspace of $H^1(D)$ consisting of the functions with vanishing trace on ∂D, then we define the space $L^p(\Omega) \otimes H_0^1(D)$,

$$L^p(\Omega) \otimes H_0^1(D) := \left\{ v : \Omega \longrightarrow H_0^1(D);\ \int_{\Omega} \|v\|^p_{H_0^1(D)} d\mathbb{P} < \infty \right\}.$$