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Numerical validation of an homogenized interface model ☆

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ABSTRACT

The aim of this paper is to numerically validate the effectiveness of a matched asymptotic expansion formal method introduced in a pioneering paper by Nguetseng and Sánchez Palencia (1985) [1] and extended in Geymonat et al. (2011) [2,3]. Using this method a simplified model for the influence of small identical heterogeneities periodically distributed on an internal surface to the overall response of a linearly elastic body is derived. In order to validate this formal method a careful numerical study compares the solution obtained by a standard method on a fine mesh to the one obtained by asymptotic expansion. We compute both the zero and the first order terms in the expansion. To efficiently compute the first order term we introduce a suitable domain decomposition method.

1. Introduction

For various researches and applications in fields spanning the mathematical, physical and engineering communities, there is great interest in efficient and mathematically rigorously justified numerical approximations of the solutions of boundary value problems where the domain and/or the coefficients have a large number of heterogeneities. Several such situations were investigated:

(i) The heterogeneities have a characteristic size which is a power of ε and are periodically distributed in *the whole volume*; the ratio between the size of the period and the size of the entire structure is traditionally denoted ε. In order to obtain numerically efficient (i.e., precise and not too computationally demanding) simplified models it is now classical to use an homogenization method. An advantage of the homogenization methods is that they have been developed and fully mathematically justified for many different geometrical and/or mechanical circumstances, in particular when the ratio between the mechanical characteristics of the heterogeneities and of the surrounding material can depend on ε [4–7].

(ii) Another situation of interest for the engineering community arises when two bodies are pasted together using a*thin layer* of another medium whose transversal depth has a ratio ε to the whole structure. Many numerically efficient simplified models have been constructed and mathematically fully justified for various mechanical and /or geometrical properties (see e.g., [8–11]).

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(iii) A situation that is new, even if there are similarities with the two previous ones, arises when the thin layer is generated by heterogeneities with characteristic size of order ε. These heterogeneities are periodically distributed, with period ε, on *an internal surface* ω. This case is considered in the present work.

It is obvious that in this case the use of standard finite element methods rapidly becomes very expensive and thus it is mandatory to find numerically efficient and, if possible, mathematically justified models to find both global (i.e., on the whole structure) and local (i.e near the heterogeneities) behavior. For heterogeneities which are holes, Nguetseng and Sánchez Palencia have proposed, for the local behavior, in a pioneering research, [1] to use a *formal multi-scale matched asymptotic expansion*. The initial problem is replaced by a set of new ones for which the layer of heterogeneities becomes a surface ω on which particular non-homogeneous jump conditions are defined. In the recent years this formal method has been used again, mainly for the global behavior, in different geometrical situations and has been generalized in order to consider not only holes but also elastic inclusions [12–15].

The *first objective* of this paper is to *numerically validate* a recently proposed variant of the multi-scale matched asymptotic expansion method [2,3] that can be successfully applied both for the global and the local behavior. In Section 2, we recall this formal method adapted to the case where different types of heterogeneities are considered. Specifically we *assume* that the displacement and stress fields admit two asymptotic expansions, one far from the heterogeneities (the outer one) another one close to them (the inner one). The *formal construction* of the outer and inner problems allows to define the transmission coefficients across the surface. As in [1] we show that the order 0 outer problem is independent of the heterogeneities. For the first order outer problem, that was not studied in [1], since the authors were interested by a local behavior, the transmission coefficients are given by several elementary inner problems posed at the so-called microscopic level, i.e., on a representative cell. These coefficients depend on the geometry and the nature of the heterogeneities (holes, highly contrasted materials). The outer problems are usually called the macroscopic problems. The *numerical validation* is described in Section 4 on a 2D test problem. The objective of this numerical validation is to study, in an appropriate norm, the error between the *reference solution* and the *numerical solution* obtained by asymptotic expansions both for zero order and first order. Since the exact solution is not known the *reference solution* is the *numerical solution* of the original problem obtained using standard finite elements on a very fine mesh. The mesh-size h has to be fine enough so that the approximation error be negligible. It is essential to study the evolution of this error when ε decreases. In Section 4.2 we have evaluated, for the coefficients appearing in the transmission conditions, the sensitivity with respect to the different parameters of the cell problems used to compute them.

The *second objective* is to give an efficient numerical algorithm which implements the computation of the macroscopic outer approximation obtained trough the zero and the first order terms. Thus in Section 3 we introduce a domain decomposition type algorithm to numerically solve the first order problem. The novelty of the situation is due to the fact that on the surface ω one has non-homogeneous transmission conditions *both for the displacements and for the normal stresses*.

We conclude the numerical validation in Section 4.5 with the comparison of the distribution of the stresses around the heterogeneities in the case of holes and of elastic inclusions. This comparison demonstrates the numerical efficiency for the local behavior, of our variant [2,3] of the multi-scale matched asymptotic expansion method.

2. Asymptotic model

2.1. The problem

Let us consider a three dimensional structure Ω (an open domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$) containing small identical heterogeneities periodically distributed along a surface ω that we assume for simplicity contained in the plane $x_1 = 0$ (see Fig. 1). We assume that ω is contained in the union of $\mathcal{N}(\varepsilon) \approx \frac{area(\omega)}{area(\widehat{Y})}\varepsilon^{-2}$ sets $\varepsilon\widehat{Y}$, where $\widehat{Y} \subset \mathbb{R}^2$ is the basis of a periodic planar net (we take for simplicity $\widehat{Y} =]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},\frac{1}{2}[$). Let I be a non-empty 3D domain contained in $\mathbb{R}\times\widehat{Y}$ with smooth

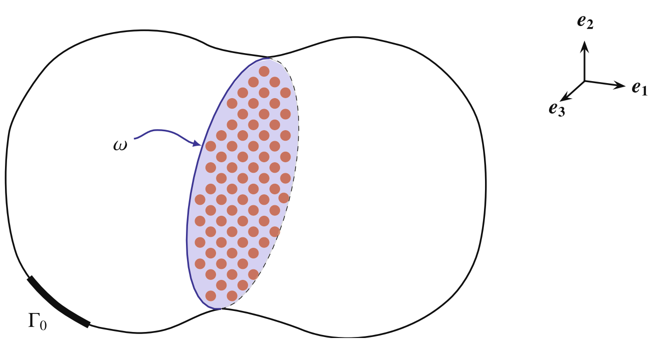

Fig. 1. The structure with the layer of heterogeneities.