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A stabilised Petrov–Galerkin formulation for linear tetrahedral elements in compressible, nearly incompressible and truly incompressible fast dynamics

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Abstract

A mixed second order stabilised Petrov–Galerkin finite element framework was recently introduced by the authors (Lee et al., 2014) [46]. The new mixed formulation, written as a system of conservation laws for the linear momentum and the deformation gradient, performs extremely well in bending dominated scenarios (even when linear tetrahedral elements are used) yielding equal order of convergence for displacements and stresses. In this paper, this formulation is further enhanced for nearly and truly incompressible deformations with three key novelties. First, a new conservation law for the Jacobian of the deformation is added into the system providing extra flexibility to the scheme. Second, a variationally consistent Petrov–Galerkin stabilisation methodology is derived. Third, an adapted fractional step method is presented for both incompressible and nearly incompressible materials in the context of nonlinear elastodynamics. For completeness and ease of understanding, these three improvements are presented both in small and large strain regimes, studying the eigen-structure of the resulting systems. A series of numerical examples are presented in order to demonstrate the robustness of the enhanced methodology with respect to the work previously published by the authors.

Keywords: Fast dynamics; Petrov–Galerkin; Fractional step; Incompressible; Locking; Geometric conservation law

1. Introduction

Classical displacement-based finite element formulations [1–7] are typically employed in industry when simulating complex engineering problems. For these applications, linear tetrahedral elements tend to be preferred when dealing with complex three dimensional geometries, due to the maturity of the existing unstructured mesh generators. However, this methodology presents a number of well-known shortcomings.

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First, reduced order of convergence for derived variables (i.e. second order for displacements but first order for stresses), requiring some form of stress recovery procedure if these are of interest [8,9]. Second, the performance of these formulations in bending dominated scenarios can be very poor [10,11] yielding unacceptable results. Third, the presence of numerical instabilities in the form of volumetric locking, shear locking and spurious hydrostatic pressure fluctuations [12,13] when large Poisson's ratios are used. This aspect is particularly relevant in the context of biomedical modelling. Fourth, from the time discretisation point of view, Newmark-type methods [14] have a tendency to introduce high frequency noise, especially in the vicinity of sharp spatial gradients and accuracy is degraded once numerical artificial damping is employed [15–19]. These schemes are thus not desirable for shock dominated problems.

Significant efforts have been undertaken to develop effective linear tetrahedral formulations for nearly incompressible solids. Multi-field Fraeijs de Veubeke–Hu–Washizu (FdVHW) type variational principles [20] are among them, where independent kinematic descriptions are used for the volumetric and deviatoric components of the deformation. The conventional mean dilatation method [21] is a particular case of Selective Reduced Integration, where the volumetric deformation is suitably underintegrated [15]. Unfortunately, the mean dilatation approach cannot be employed with linear tetrahedrals and authors resort to some form of projection to reduce the number of volumetric constraints [21–29]. Alternatively, high order interpolation approaches can be used [30,31]. However, the increase in the number of integration points can drastically reduce the computational efficiency of these schemes in comparison with low order approaches, specially when complex constitutive laws must be modelled (e.g. visco-elasticity, visco-plasticity).

A family of nodally integrated tetrahedral elements was formulated in [32], where the volumetric strain energy functional was approximated through averaged nodal pressures. However, the resulting approach was reported to behave poorly in bending dominated scenarios. To overcome this difficulty, reference [11] proposed the nodal based uniform strain tetrahedral element by applying a nodal averaging process to the whole small strain tensor. Reference [33] extended this application to the large strain regime with the idea of employing both a nodal average Jacobian and a nodal average deformation gradient in the calculation of the stress tensor. As reported in [34–37], the resulting formulation suffers from artificial mechanisms similar to hourglassing unless some form of stabilisation is used. Despite exhibiting very good behaviour in terms of displacements, this class of averaged nodal strain tetrahedral formulations tend to exhibit non-physical hydrostatic pressure fluctuations [34,38].

In parallel, in reference [39], a stabilised Petrov–Galerkin (PG) formulation by using the Galerkin Least Squares (GLS) approach is first introduced for the analysis of the Stokes problem, with equal order of interpolation for velocity and pressure. The formulation circumvents the Ladyzenskaya–Babuska–Brezzi (LBB) condition [40,41], ensuring numerical stability and optimal convergence.

At present and to the best of our knowledge, most of the proposed schemes for linear tetrahedral elements are restricted to elastostatics [34,42–45]. The development of an effective linear tetrahedral formulation in the range of fully and nearly incompressible large strain dynamics remains an open issue.

The aim of this paper is to improve the robustness and effectiveness of the stabilised Petrov–Galerkin (PG) mixed finite element framework recently presented in [46], extending its applicability to the range of fully and nearly incompressible materials. This mixed methodology is formulated in the form of a system of first order conservation laws [10,46–48], where the linear momentum $\boldsymbol{p}$ and the deformation gradient tensor $\boldsymbol{F}$ of the system are regarded as the main conservation variables of this mixed $\boldsymbol{p}$-$\boldsymbol{F}$ approach. In [46], a robust and stable PG implementation is presented, derived with the help of the Variational Multi-Scale (VMS) method [49–52]. Unfortunately, in the case of extreme deformations in the incompressible limit (i.e. refer to twisting column example in Section 4.4 of this paper), the $\boldsymbol{p}$-$\boldsymbol{F}$ formulation lacks robustness.

With this in mind, this mixed PG formulation is first enhanced by introducing a new conservation law for the Jacobian J of the deformation (volumetric strain in the small strain regime), also known as a Geometric Conservation Law [53]. The volumetric stress component appearing in the conservation of linear momentum equation is then evaluated from this new conservation law, providing more flexibility and robustness to the scheme.

For computational efficiency, the enhanced $\boldsymbol{p}$-$\boldsymbol{F}$-$\boldsymbol{J}$ formulation is implemented in conjunction with an explicit time integrator, where the time step size is controlled through the Courant–Friedrichs–Lewy number [54] by the volumetric wave speed c_p. In the incompressibility limit, c_p can reach very high values leading to a very