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On the Virtual Element Method for three-dimensional linear elasticity problems on arbitrary polyhedral meshes

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Abstract

We explore the recently-proposed Virtual Element Method (VEM) for the numerical solution of boundary value problems on arbitrary polyhedral meshes. More specifically, we focus on the linear elasticity equations in three-dimensions and elaborate upon the key concepts underlying the first-order VEM. While the point of departure is a conforming Galerkin framework, the distinguishing feature of VEM is that it does not require an explicit computation of the trial and test spaces, thereby circumventing a barrier to standard finite element discretizations on arbitrary grids. At the heart of the method is a particular kinematic decomposition of element deformation states which, in turn, leads to a corresponding decomposition of strain energy. By capturing the energy of linear deformations exactly, one can guarantee satisfaction of the patch test and optimal convergence of numerical solutions. The decomposition itself is enabled by local projection maps that appropriately extract the rigid body motion and constant strain components of the deformation. As we show, computing these projection maps and subsequently the local stiffness matrices, in practice, reduces to the computation of purely geometric quantities. In addition to discussing aspects of implementation of the method, we present several numerical studies in order to verify convergence of the VEM and evaluate its performance for various types of meshes.

Keywords: Virtual Element Method; Mimetic Finite Difference; Polyhedral meshes; Polytopes; Voronoi tessellations

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1. Introduction

The development of discretization methods for solving three-dimensional boundary value problems on general polyhedral meshes has recently received considerable attention in the numerical analysis literature. One driving force behind this trend is the difficulty associated with mesh generation for complex or evolving domains, for which the use of arbitrarily-shaped elements can provide much needed flexibility [1]. For example, a simple embedding strategy consisting of carving out the problem domain out of a structure background grid, produces polyhedral elements at the boundary [2]. Mesh refinement and coarsening in adaptive schemes can also be handled with greater ease if the

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analysis method allows for the presence of elements with general geometries [3]. In addition to advantages in mesh generation and adaptation, polyhedral discretizations can deliver improved performance in some applications. For example, as discussed in [4], polyhedral meshes can achieve the same level of accuracy in flow simulations compared to their simplicial counterparts but with far fewer number of cells and unknowns.

With regards to the type of discretization method, finite volume methods based on polyhedral cells have reached a level of maturity in fluid dynamic simulations, as evidenced by their availability and use in commercial software [5,6]. Mimetic finite difference (MFD) methods, capable of handling general three-dimensional meshes, are also the subject of active research and have been successfully applied to diffusion, elasticity, and fluid flow problems (see, for example, [7–11]). The extension of finite element methods in this arena, however, has been relatively slow, despite the availability of special interpolation functions in the literature. This is, in part, due to the fact that these interpolants are subject to restrictions on the geometry of admissible elements (e.g., convexity, maximum valence count) and can be sensitive to geometric degeneracies. More importantly, calculating these functions and their gradients are often prohibitively expensive. Numerical evaluation of weak form integrals, with sufficient accuracy, poses yet another challenge due to the non-polynomial nature of these functions as well as the arbitrary domain of integration[2] [12]. To mention a few approaches in the literature aiming to overcome these barriers, we point to the work by Rashid and co-workers [13,2,3], who have developed elements based on non-conforming polynomial or piecewise polynomial basis functions that are tolerant of degeneracies. More recently, harmonic basis functions have been considered by [14,15] with particular attention to alleviating the cost of their computation and integration. Other works include constructions based on natural element [16], non-Sibson [17], and mean value coordinates [18], the smoothed finite element method [19], and the extension of the so-called mean-quadrature approach to polyhedral grids [20].

In this work, we focus on the recently-developed Virtual Element Method (VEM) that addresses some of the above-mentioned challenges facing finite element schemes [21–24]. As with finite elements, VEM is a Galerkin scheme with an underlying approximation space defined according to a partition (mesh) of the domain. However, it is distinguished from classical finite elements in that it does not require the computation of the interpolation functions in the interior of the elements. One goal of the present work is to break down and elaborate upon the core mathematical concepts underlying VEM within the context of linear elasticity boundary value problems. The key to the success of the method is a consistent approximation to the elemental strain energy that is exact for the linear deformations without requiring volumetric integration of the basis functions. What enables this approximation is a set of local projection maps that appropriately split up the element deformation into its polynomial and non-polynomial components. In the case of the first-order VEM formulation, where the degrees of freedom are associated with the vertices of the elements, two projection maps, associated with rigid body motion and constant strain deformations, respectively, are used to achieve this kinematic decomposition. As we shall discuss, these projection maps can be beneficial even for finite element schemes when one has access to interpolation functions. We should note that while VEM provides the general recipe for extension to higher-order and higher-continuity polyhedral elements (see, for example, [25]), this significant technology may be hidden in this paper as we will limit the discussion, for the sake of clarity, to the first-order formulation. We refer to [22] for a discussion of arbitrary order VEM for both compressible and incompressible elasticity in two dimensions.

The other task undertaken here is to discuss, in detail, aspects of the implementation of the method for general polyhedral meshes. To this effect, we will derive explicit expressions for the element stiffness matrix and discrete representation of the element projection maps. In addition to two matrices containing special arrangements of coordinates of the element vertices, we encounter two matrices that require calculation of surface integrals of the basis functions over the element boundary. These quantities also reduce to geometric information of the faces (centroids, areas, etc.), if either the approximation spaces are based on interpolants derived by [26] or if a consistent nodal quadrature rule is used. While the connection between the MFD method and VEM has been established in the original papers on VEM, the discussion here further elucidates this relationship and illustrates how the Galerkin framework with an underlying approximation space serves as a vehicle for constructing a method that is ultimately geometric in nature. Finally, we note that the recent work [24] also discusses practical aspects of implementation of VEM for second-order elliptic problems in two and three dimensions.

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[2] By contrast, classical finite elements feature interpolation functions that are either polynomials or images of polynomials and numerical integration is carried out by means of a mapping to a fixed parent domain.