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One-dimensional nonlocal and gradient elasticity: Assessment of high order approximation schemes

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Abstract

We investigate the application and performance of high-order approximation techniques to one-dimensional nonlocal elastic rods. Governing equations and corresponding discrete forms are derived for the integro-differential formulation proposed by Eringen and the laplacian-based strain gradient formulation developed by Aifantis and coworkers. Accuracy and convergence rate of the numerical solutions obtained with Lagrange, Hermite, B-spline finite elements and C^∞ generalized finite elements are assessed against the corresponding analytical solutions.

Keywords: Nonlocal elasticity; Strain gradient elasticity; FEM; B-splines; C^∞ GFEM

1. Introduction

Effective modeling of nonlocal problems is a challenging issue in computational mechanics. In this context, high order approximation schemes are often required. Here, we compare standard and high order approximation schemes with a high degree of continuity in the analysis of one-dimensional nonlocal elasticity boundary value problems. In particular, we consider problems whose solution fields present local area of high gradients at the boundary or within the domain.

It is well known that classical continuum mechanics fails to predict deformation phenomena at the nanoscale due to the absence of an internal material length scale in the constitutive law. Therefore, nonclassical formulations have been proposed to model size-dependent problems where the effect of material microstructure and long-range interatomic forces becomes predominant. In this context, one-dimensional

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nonlocal formulations have been used to describe the mechanical response of nanostructures such as nanotrusses and carbon nanotubes [1,2].

The finite element method is the *de facto* choice for the analysis of systems which show a size-dependent mechanical response [3–6], for the analytical solution of these problems is usually difficult even in the most simple cases. Nevertheless, several issues arise from the implementation of either computationally intensive algorithms or highly continuous approximation schemes when derivatives of the strain field are included in the constitutive equations. In fact, the numerical solution of nonlocal elastic problems poses great challenges since most of the approaches in the literature show limitations. Equally important, the finite element analysis of size-dependent systems depends on the constitutive equations employed to represent the nonlocal medium. Among others, Eringen [7] suggested an integro-differential formulation where the stress at a given point is made a weighted function of the strain at surrounding points. The corresponding finite element formulation leads to significant computational effort since the stiffness matrix reflects the nonlocality of the material [8] and looses sparsity. Aifantis and coworkers extended the classical elastic constitutive equations with the Laplacian of the strain tensor [9], consequently increasing the continuity requirements for the approximation schemes [10] which need to be C^1-continuous or higher.

In this work, non-conventional approximation techniques are tested on the aforementioned nonlocal elastic models and compared against traditional approaches. We consider B-spline finite elements and C^∞ generalized finite elements (C^∞ GFEM) along with classical Lagrange and Hermite discretization techniques. The first method is widely used to approximate smooth and free-form geometries [11] and employs high-order piecewise polynomial basis functions. This technique can achieve a high degree of continuity through the so-called *k*-refinement technique [12] as described in Section 2.1. Furthermore, B-spline finite elements have been the subject of several publications, especially because of the interest in the Isogeometric Analysis [12–15]. The second approximation scheme uses a C^∞ partition of unity (PoU) and polynomial enrichments to build arbitrarily smooth basis functions [16] as detailed in Section 2.2.

The main objective of this work is to present benchmark studies aimed at shedding some light on the assessment of the accuracy of the aforementioned approximation schemes. Starting from the governing equilibrium equations of the tensile rod, their discrete form is derived for the integro-differential formulation and the strain gradient model described in Sections 3 and 4, respectively. We discuss the main features which distinguish these approaches and characterize implementation and continuity requirements on the discretization. We then consider some practical applications. In particular, the case study of a tensile rod with constant stress is used to compare the numerical results obtained with the two constitutive models employing the approximation schemes previously described. The performance of the approximation schemes is further assessed by means of two other examples: a homogeneous tensile rod under the action of a body force with high gradient employing the integro-differential formulation, and a strain gradient bimaterial rod with constant stress. The MATLAB® scripts used in the benchmark studies are freely available for download at the corresponding author's web site.

To the best of our knowledge, this paper represents the first attempt to compare the accuracy of classical and novel approximation techniques in the solution of nonlocal problems. In particular, we discuss the numerical results obtained from the integro-differential and the strain-gradient approaches in great detail. Furthermore, for the first time within the framework of the finite element method, high order approximations are used in the discrete equations derived from the integro-differential approach. It is not possible to say *a priori* which method would be the most suitable to solve these equations. Hence, this work may provide significant insight into the applicability of high order methods in size-dependent problems which are for instance very useful for developers of nonlocal FE models for nanowires and nanotubes [17–21].

2. Approximation schemes

In the finite element method analysis of one-dimensional elasticity problems, the displacement field $u(x)$ and the corresponding strain field $\varepsilon(x)$ are approximated at the element level through

$$u^e(x) \simeq \mathbf{N}^e(x)\,\mathbf{u}^e \quad \text{and} \quad \varepsilon^e(x) \simeq \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}x}\,\mathbf{u}^e = \mathbf{B}^e(x)\,\mathbf{u}^e, \tag{1}$$