Comput. Methods Appl. Mech. Engrg. 273 (2014) 19–36

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

# A fast and faithful collocation method with efficient matrix assembly for a two-dimensional nonlocal diffusion model

Hong Wang [a,*], Hao Tian [b,1]

[a] Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
[b] School of Mathematics, Shandong University, Jinan, Shandong 250100, China

ARTICLE INFO

*Article history:*
Received 28 July 2013
Received in revised form 15 January 2014
Accepted 28 January 2014
Available online 7 February 2014

*Keywords:*
Collocation method
Dense matrix
Discontinuous problems
Fast method
Nonlocal diffusion

ABSTRACT

The peridynamic theory provides an appropriate description of the deformation of a continuous body involving discontinuities or other singularities, which cannot be described properly by the classical theory of solid mechanics. However, the operator in the peridynamic theory is nonlocal, so the resulting numerical methods generate dense or full coefficient matrices which require $O(N^2)$ of memory where $N$ is the number of unknowns in the discretized system. Gaussian types of direct solvers, which were traditionally used to solve these problems, require $O(N^3)$ of operations. Furthermore, due to the singularity of the kernel in the peridynamic model, the evaluation and assembly of the coefficient matrix can be very expensive. Numerous numerical experiments have shown that in many practical simulations the evaluation and assembly of the coefficient matrix often constitute the main computational cost! The significantly increased computational work and memory requirement of the peridynamic model over those for the classical partial differential equation models severely limit their applications, especially in multiple space dimensions.

We develop a fast and faithful collocation method for a two-dimensional nonlocal diffusion model, which can be viewed as a scalar-valued version of a peridynamic model, without using any lossy compression, but rather, by exploiting the structure of the coefficient matrix. The new method reduces the evaluation and assembly of the coefficient matrix by $O(N)$, reduces the computational work from $O(N^3)$ required by the traditional methods to $O(N\log^2 N)$ and the memory requirement from $O(N^2)$ to $O(N)$. Numerical results are presented to show the utility of the fast method.



## 1. Introduction

The classical theory of solid mechanics assumes that all internal forces in a body act through a zero distance, which leads to mathematical models described by partial differential equations. These models do not provide a proper description of problems with spontaneous formation of discontinuities. The peridynamic theory was proposed as a reformulation of solid mechanics [5,13,18,27,29–32,38,39], which leads to a nonlocal or integro-differential framework that does not explicitly involve the notion of deformation gradients.

* Corresponding author. Tel.: +1 18037774321.
*E-mail address:* hwang@math.sc.edu (H. Wang).
[1] Current address: Beijing Computational Science Research Center, Beijing, China.

http://dx.doi.org/10.1016/j.cma.2014.01.026

The numerical solution of integral or boundary integral equations by Galerkin finite element methods or collocation methods can at least date back to 1970s [12,19,20]. In the context of peridynamic models or their scalar-valued version nonlocal diffusion models, different numerical methods, including Galerkin finite element (or weak form) methods, discontinuous Galerkin finite element methods, collocation (or strong form) methods, and meshfree methods, have been developed and analyzed [3,8,10,14,21,23,26,28]. In contrast to those for classical differential equation models for solid mechanics, the numerical methods for peridynamic or nonlocal diffusion models, as those for integral or boundary integral equations, generate dense or full coefficient matrices which require $O(N^2)$ of memory where $N$ is the number of unknowns in the discretized system. Gaussian types of direct solvers, which were traditionally used to solve these problems, require $O(N^3)$ of operations. Furthermore, due to the singularity of the kernel in these models, the evaluation and assembly of the coefficient matrix can be very expensive. Numerous numerical experiments have shown that in many practical simulations the evaluation and assembly of the coefficient matrix often constitute the main computational cost! The significantly increased computational work and memory requirement of these models over those for the classical partial differential equation models severely limit the applications of peridynamics, especially in multiple space dimensions.

Much effort has been taken to reduce the computational cost and memory requirement of the numerical methods for peridynamic or nonlocal diffusion models. A widely used simplification in the numerical simulation of peridynamic or nonlocal diffusion methods is to assume that the horizon of the material $\delta = O(h)$ [8], where $h$ is the mesh size. Under this assumption, the structure of the coefficient matrix of the corresponding numerical methods for peridynamic or nonlocal diffusion models is similar to that of high-order numerical methods for partial differential equation models. And so are the memory requirement and computational cost of the numerical methods for a simplified nonlocal diffusion model. However, the physical relevance of the simplification $\delta = O(h)$ is not very clear as the horizon $\delta$ represents a material property that is independent of a computational mesh size. Another indication of this inconsistency is that the optimal-order error estimate for Galerkin finite element methods of full peridynamic or nonlocal diffusion models becomes one-order suboptimal compared to those of the simplified peridynamic or nonlocal diffusion model (refer to Section 4.2).

This paper concerns numerical methods for nonlocal diffusion models, which involve scalar fields. The proposed methods can be extended to models involving vector fields, such as peridynamic models. The objective of this paper is not to come up with another discretization scheme for a nonlocal diffusion model, but rather to develop a fast and faithful solution technique for an existing piecewise-bilinear collocation method for a (non-simplified) nonlocal diffusion model in two space dimensions. We prove that for any translation-invariant kernel function and any neighborhood, the corresponding stiffness matrix has certain block-Toeplitz-Toeplitz-block structure. Consequently, the fast method (i) reduces the evaluation and assembly of the stiffness matrix from $O(N^2)$ entries to $O(N)$ entries and the memory requirement from $O(N^2)$ entries to $O(N)$ entries; (ii) reduces the computational cost of the inversion of the numerical scheme to $O(N \log N)$ per iteration. The exploration of Toeplitz and circulant matrice structures of the stiffness matrices for the reduction of computational work and memory requirement has been one of the techniques used in the development of fast numerical methods for partial differential equations or integral equations [2,7,15]. In the past few years one of the authors codeveloped fast finite difference methods for space-fractional diffusion equations by proving that the corresponding stiffness matrix can be decomposed as a sum of diagonal multiplied by Toeplitz matrices [35–37]. Recently, the authors developed a fast Galerkin finite element method for a one-dimensional nonlocal diffusion model problem [34]. In this paper we develop a fast and faithful piecewise-bilinear collocation method for a two-dimensional nonlocal diffusion model problem by exploiting a block-Toeplitz-Toeplitz-block structure of the stiffness matrix of the numerical method. Hence, the resulting fast method is not lossy and retains the stability and accuracy of the underlying bilinear collocation method while significantly reducing its computational cost and memory requirement.

The rest of the paper is organized as follows. In Section 2 we present a nonlocal diffusion model in two space dimensions and a bilinear collocation method for the model problem. In Section 3 we develop a fast and faithful collocation method with an efficient matrix assembly and storage, not by employing any lossy compression, but rather, by exploiting a block-Toeplitz-Toeplitz-block structure of the stiffness matrix for any translation-invariant kernel function in the model. In Section 4 we discuss possible extensions of the fast bilinear collocation method to other problems and the relation of the collocation method with Galerkin finite element method. In Section 5 we conduct numerical experiments to investigate the performance of the fast collocation method. In Section 6 we draw some concluding remarks and discuss future directions.

## 2. A nonlocal diffusion model and its bilinear collocation approximation

We begin this section by considering a two-dimensional nonlocal diffusion model to be numerically solved in this paper. Then we present its bilinear collocation approximation based upon which we develop a fast and faithful solution method.

### 2.1. A nonlocal diffusion model

A linear steady-state nonlocal diffusion model in the plane is given by the following integral equation with the prescribed boundary condition [17,26,27]