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An immersed boundary method for fluids using the XFEM and the hydrodynamic Boltzmann transport equation

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ABSTRACT

This paper presents a stabilized finite element formulation of the hydrodynamic Boltzmann transport equation (HBTE) to predict nearly incompressible fluid flow. The HBTE is discretized with Hermite polynomials in the velocity variable, and a streamline upwind Petrov–Galerkin formulation is used to discretize the spatial variable. A nonlinear stabilization scheme is presented, from which a simple linear stabilization scheme is constructed. In contrast to the Navier–Stokes (NS) equations, the HBTE is a first order equation and allows for conveniently enforcing Dirichlet conditions along immersed boundaries. A simple and efficient formulation for enforcing Dirichlet boundary conditions is presented and its accuracy is studied for immersed boundaries captured by the extended finite element method (XFEM). Numerical experiments indicate that both the linear and non-linear stabilization methods are sufficiently accurate and stable, but the linear formulation reduces the computational cost significantly. The accuracy of enforcing boundary conditions is satisfactory and shows second order convergence as the mesh is refined. Augmenting the boundary condition formulation with a penalty term increases the accuracy of enforcing the boundary condition constraints, but may degrade the accuracy of the global solution. Comparisons with results of a single relaxation time lattice Boltzmann method show that the proposed finite element method features greater robustness and lesser dependence of the computational costs on the level of mesh refinement.

1. Introduction

Immersed boundary methods are attractive when the geometry is difficult to mesh, and for applications with dynamically evolving geometry including multi-phase flows and topology optimization. Dirichlet boundary conditions in traditional finite element methods are conveniently imposed by specifying nodal values. Prescribing the value of a state variable within an element is more difficult because the state value is a function of several degrees of freedom. This situation occurs frequently when complex geometries are represented with, for example, the extended finite element method (XFEM) or iso-geometric finite element methods.

The flexibility of immersed boundary methods has attracted significant attention concerning the treatment of Dirichlet conditions, see Stenberg [1] and Lew and Buscaglia [2] for an overview. Imposing Dirichlet boundary conditions along

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immersed boundaries for second order partial differential equations (PDEs) is not straight-forward. The function space of Lagrange multiplier methods needs to satisfy the "inf–sup" condition to converge optimally [3], and stabilized Lagrange multiplier methods and Nitsche methods include parameters that can cause ill-conditioning of the linear system if not chosen properly.

More recently, immersed boundary finite element methods have been applied to the Navier–Stokes (NS) equations. For example, Gerstenberger and Wall [4] and Kreissl and Maute [5] studied a stabilized Lagrange multiplier-like method for the NS equations by adding an auxiliary stress field that approximates the stress required to achieve the physical behavior at the boundary. The root cause of the difficulty in imposing immersed boundary conditions for the NS equations is the second order viscous term. Bypassing this issue is one motivation to describe nearly incompressible fluid flows by the hydrodynamic Boltzmann transport equation (HBTE), which is a first order equation and allows convenient enforcement of Dirichlet boundary conditions.

The HBTE is a kinetic theory approach to fluid dynamics, whereas the NS equations are derived from the conservation of momentum in a continuum of fluid. The HBTE describes the time evolution of a particle distribution, $f(\mathbf{x},\xi,t)$, as a function of the spatial and velocity variables. The form of the continuous HBTE under a Bhatnaggar Gross Krook relaxation time approximation [6] is:

$$\frac{\partial f(\mathbf{x},\xi,t)}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f(\mathbf{x},\xi,t) = -\frac{f(\mathbf{x},\xi,t) - f^{eq}(\mathbf{x},\xi,t)}{r}, \tag{1}$$

where $\mathbf{x}$ represents the spatial variable, ξ represents the velocity variable, t is the time, $f^{eq}(\mathbf{x},\xi,t)$ is the equilibrium distribution, and r is the relaxation time. The right hand side of the continuous HBTE (1) is referred to as the collision operator. The continuous HBTE has been shown to recover the NS equations [7], but includes the flexibility to represent finite Knudsen number flows [8]. The focus of this paper is on continuum flows; finite Knudsen number flows will be the subject of future research.

A growing portion of the computational fluid dynamics community has focused on the lattice Boltzmann method (LBM) over the past two decades. A general overview of the LBM is provided by Yu et al. [9]. The LBM is an explicit finite difference discretization of the hydrodynamic Boltzmann transport equation which leads to an algorithmically simple computational procedure. While the LBM enjoys several numerical advantages, including few floating point operations per lattice update and easy parallelization, there are several disadvantages to this popular method. By construction the LBM operates on structured meshes with explicit time integration. In contrast to finite element methods, the LBM lacks a mathematical formalism for unstructured meshes, and local mesh refinement is more complex because the model parameters depend on the mesh spacing, see for example [10]. The explicit time integration limits the time step size according to the Courant–Friedrichs–Lewy (CFL) condition, which can be increasingly restrictive as the computational mesh is refined. The LBM inherently satisfies the CFL condition, but nonlinear instabilities may arise if the computational grid is too coarse for a given problem. The mesh should be sufficiently refined to achieve stable values for the relaxation time and the lattice velocity. However, the computational time required grows on the order of $O(M^4)$ in three dimensions, where M is the number of lattice cells that span the characteristic length. Finally, accurate boundary condition enforcement schemes are difficult to develop, particularly for curved boundaries [11].

The limitations of the traditional LBM have created interest in applying standard discretization techniques including finite difference [12–15], finite volume [16–22], and finite element techniques. There has been an increase in research on finite element methods for the HBTE in the last decade. Lee and Lin [23] presented a characteristic Galerkin finite element method, and Li et al. [24,25] employed a least squares finite element method. Several other authors have investigated discontinuous Galerkin schemes [26–30].

Generalized numerical methods have three major advantages over the LBM. First, implicit or explicit time integration schemes with an arbitrary order of accuracy can be applied. Second, the numerical stability can be enhanced [25]. Third, the velocity variable can be discretized with any suitable interpolation scheme and represented with arbitrary accuracy. The most common two dimensional LBM uses nine discrete ordinates in the velocity space, which is for algorithmic simplicity and is not a necessity to capture the physical behavior. Tölke et al. [15] followed the approach of Grad [7] and discretized the velocity space with Hermite polynomials and included only six Hermite coefficients. Six Hermite polynomial coefficients are the minimum number of degrees of freedom necessary to recover the NS equations for nearly incompressible flow (i.e., low Mach number flow) in two dimensions. This discretization of the velocity variable results in a unique relationship between the degrees of freedom and the macroscopic physical quantities [15] and simplifies the application of boundary conditions. The boundary conditions appear as typical Dirichlet conditions or as linear constraints on the state variables, which for the purpose of this paper will be referred to as Dirichlet conditions. The velocity variable can be resolved with more degrees of freedom to describe rarefied or high Mach number flow [26,31].

It is necessary in finite element methods to stabilize the advection term in the HBTE to prevent spurious spatial oscillations in the state variable field. The streamline upwind Petrov–Galerkin (SUPG) stabilized finite element method [32] has been applied to a wide class of advection dominated problems [33–35]. A variation called the subgrid-scale finite element method was applied to the radiative transport equation [36] with discrete ordinates in the velocity variable. To the authors' knowledge, an SUPG stabilized formulation for the hydrodynamic Boltzmann transport equation is yet to be formulated for either discrete ordinate or Hermite polynomial discretized velocity spaces. The difficulty is in developing the matrix of stabilization parameters, commonly referred to as $\boldsymbol{\tau}$. The collision term complicates the definition of $\boldsymbol{\tau}$ for any velocity space