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Ideal minimal residual-based proper generalized decomposition for non-symmetric multi-field models – Application to transient elastodynamics in space-time domain

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ABSTRACT

It is now well established that separated representations built with the help of proper generalized decomposition (PGD) can drastically reduce computational costs associated with solution of a wide variety of problems. However, it is still an open question to know if separated representations can be efficiently used to approximate solutions of hyperbolic evolution problems in space-time domain. In this paper, we numerically address this issue and concentrate on transient elastodynamic models. For such models, the operator associated with the space-time problem is non-symmetric and low-rank approximations are classically computed by minimizing the space-time residual in a natural L_2 sense, yet leading to non optimal approximations in usual solution norms. Therefore, a new algorithm has been recently introduced by one of the authors and allows to find a quasi-optimal low-rank approximation *a priori* with respect to a target norm. We presently extend this new algorithm to multi-field models. The proposed algorithm is applied to elastodynamics formulated over space-time domain with the Time Discontinuous Galerkin method in displacement and velocity. Numerical examples demonstrate convergence of the proposed algorithm and comparisons are made with classical *a posteriori* and *a priori* approaches.

1. Introduction

Standard discretization techniques, such as finite element method and time integration schemes, are now sufficiently well-established to be used for the solution of industrial problems. However, applying these methods in industrial context requires ever increasing computational resources. As an example, performing one simple transient elastodynamic simulation, on a structure discretized with one million nodes and over one million time steps, requires more than 7000 Gbytes in order to store a scalar solution over the space-time domain. Such a solution can never be stored in practice and engineers have the difficult task (before performing analyses) to select particular locations in space-time domain for partially post-processing the full solution, other calculated values being lost at the end of the simulation. Also, industrial problems usually depend on many parameters and engineers need to perform as many simulations as possible in optimization or uncertainty

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quantification contexts. Then, even if the impressive amount of computational resources today available potentially allows to deal with such models, one should propose innovative methods to better exploit these resources.

Reduced order models (ROMs) can be used to address this issue. They are usually based on projection of the full order model onto low dimensional reduced bases. The most popular reduced basis in elastodynamics is the truncated modal basis due to the physical interpretation of spatial modes [23,37]. However, ROMs based on modal truncation are not efficient to simulate transient motions since eigen modes associated with a wide frequency band must be introduced in the reduced basis to get accurate results. More recently, ROMs based on proper orthogonal decomposition (POD) were proposed [7,29,28]. In this approach, the full order model in space is used to compute some solutions at particular time instants of the computational domain (called snapshots). Then, the idea is to build the reduced basis with the most dominant modes of the POD of the snapshots. This method is well-suited when reasonable comparability of model dynamics can be assumed [24]. Otherwise, many snapshots must be computed to obtain a reduced basis representative of all full order model states. POD method is called *a posteriori* since the reduced basis must be computed in a first step, based on partial knowledge of the full order model solution. In this paper, we present an *a priori* approach that allows finding a good approximation of the POD of the full space-time solution, without computing this full solution. It is based on a new algorithm recently proposed in [8].

Initially introduced in the 80's for the solution of non-linear mechanical models [30], ROMs based on PGD have been more recently generalized to high-dimensional problems [5,1,15,35]. In the last decade, lots of contributions have been published, showing that many problems solutions can be well approximated using PGD of low ranks [16]. For evolution problems [2,36], the aim is to approximate a space-time solution as a sum of products of space and time functions, that can be viewed as space-time modes. The number of space-time modes is called decomposition rank. The question is then, can we accurately approximate the solution of a given evolution problem with decomposition of low rank? For second order hyperbolic problems the question is still open (see [10] for a first application of space-time PGD on the equation of waves motion). In this paper, we numerically address this issue and concentrate on solutions of transient elastodynamic models with two dimensions in space.

To evaluate if such solutions can be well approximated with separated representations of low rank, we first need to define what is the best approximation of a given rank M. Assuming the full solution is known, we should introduce a target norm in order to measure the error between the full solution and the low rank approximations. Then, the optimal approximation of rank M is defined as a minimizer of this error in the subset of rank-M tensors. This construction is called *a posteriori* decomposition [25] since one needs to calculate and be able to store the full solution before computing its low rank approximation. When this is not possible, a more challenging task is to compute a low rank approximation *a priori*, that is without knowing the full solution. This is aim of classical PGD methods. An even more challenging task is to compute the optimal low rank approximation *a priori* and this is aim of this paper.

For linear systems of equations with non-symmetric operators, as is the case of elastodynamic problems, PGD is classically computed by minimizing the discrete residual typically using (L_2-type) Euclidean norms [3,36]. However, when operators are ill conditioned, the classical minimal residual PGD yields poorly accurate approximations with respect to usual solution norms. In elastodynamic problems, the number of space-time modes required to represent the full solution at a given accuracy is far from the optimal one. Therefore a new algorithm for non-symmetric problems has been introduced in [8] and allows finding *a priori* a quasi-optimal approximation of a given rank, with respect to a target metric. The idea is to introduce an ideal residual norm, so that minimizing this residual norm is equivalent to minimizing the error in the target norm. Notice that other variants have been also proposed in [36,12].

We presently extend the algorithm proposed in [8] to multi-field models for its application in elastodynamics. This is motivated by the fact that recent time integration schemes used to compute an accurate solution of transient elastodynamic problems, are often based on multi-field formulations [27]. Here, we formulate the elastodynamic problem with two fields (displacement and velocity) and we use the Time Discontinuous Galerkin (TDG) method [26]. This formulation allows avoiding high frequency perturbations introduced by classical time schemes (such Newmark scheme) and that can increase the number of space-time modes required to approximate the full solution at a given accuracy [10]. Moreover, for elastodynamic models with two dimensions in space (or more), unknown solutions are vector fields (displacement and velocity vectors). In such case, each component of the unknown vectors is taken as a particular field.

This paper is structured as follows: in Section 2, the optimal approximation of a given rank with respect to a target norm is defined and heuristic comparisons are made with approximations based on classical residual minimizations. In Section 3, the ideal minimal residual-based approach allowing finding a quasi-optimal approximation with respect to the target metric is described. In Section 4, we extend this approach to multi-field models and an example is given in Section 5 for the elastodynamic problem with two dimensions in space. In Section 6, space-time separated approximations of transient responses due to impact loads whose duration varies from 0.01 to 1 ms are computed. Numerical comparisons are made between *a posteriori* and *a priori* low-rank approximations, and convergence of the new algorithm is analyzed.

2. Optimality of low-rank approximations

In this section, we introduce space-time separated representations of solutions of single field models. First, we define the optimal separated approximation of a given rank with respect to a target metric. Then, we give some heuristic arguments to explain why this optimal decomposition may not be well approximated *a priori* with classical minimal residual approach if the operator associated with the considered model is ill conditioned.