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An anisotropic, fully adaptive algorithm for the solution of convection-dominated equations with semi-Lagrangian schemes

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ABSTRACT

We present in this paper an anisotropic, fully adaptive spatial–temporal algorithm for the solution of convection-dominated equations in the context of semi-Lagrangian schemes. We devise the algorithm within a finite element framework suitable for higher-order finite elements, and derive a newly proposed a posteriori error indicator which allows us to control the local or truncation error in the L^2-norm at each time step. This a posteriori error is split into temporal and spatial contributions, leading us to define an optimal time step size and an optimal triangulation, respectively. As regards the spatial adaptation, anisotropic, unstructured triangular meshes are used to capture the distinctive features of the evolving discrete solution of the governing equations. For solutions exhibiting strong anisotropies, the orientation, shape and size of the mesh triangles are provided by a metric tensor valid for linear and quadratic finite elements.

Finally, we show the capabilities of the algorithm, for linear and quadratic finite elements, by a series of two- and three-dimensional benchmarks taken from the literature, involving purely convective as well as convection-dominated problems.

1. Introduction

The number of real-world applications making use of the concept of 'adaptivity' is ever increasing. Whether temporal or spatial, the concurrence of multiple scales in a scientific or engineering problem poses a challenge to its numerical solution. Nonetheless, a thorough understanding of the physical, chemical or mathematical properties of the problem is often enough to perform a 'selective' resolution (coarse grained, fine grained) of that problem in a certain time interval (temporal adaptivity) and/or around specific regions of the domain (spatial adaptivity), which allows for huge computational savings ensuing from such an approach.

Regarding mesh adaptation, a first step is to resize the elements comprising the mesh without otherwise altering the shape or orientation of those elements. Thus 'isotropic' adaptation occurs, trying to keep the triangles (within a Finite Element context) as close to equilateral as possible (see, e.g. [1,2]). However, a wide range of applications displays 'directional features' (shock waves, jets, flames, vortices, fracture propagation, etc., to name a few) which cannot be accurately tracked by isotropic meshes: 'anisotropic' adaptation then addresses this issue by not only refining the size of the elements, but also

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modifying their shape and orientation as required, resulting in high aspect ratio elements that usually produce lower spatial error for a given tolerance even with a reduced number of elements; see for instance [3] for a detailed account of anisotropic adaptation in CFD applications. Hence, size, shape and orientation are key parameters to the distribution of the spatial error over the domain, providing the means to isotropic (size) as well as anisotropic (size, shape, orientation) adaptivity.

D'Azevedo [4] and D'Azevedo and Simpson [5] introduced the relation between an optimal anisotropic triangulation and linear interpolation error estimation for quadratic functions, which later gave rise to the concept of 'metric tensor' (see, e.g. [3,6,7]), now common in state-of-the-art, anisotropic, mesh generation codes. After that, the generalization to more general functions (not only quadratics) and norms in different Sobolev spaces was studied, and the derived anisotropic error estimate was then applied to stationary problems as can be seen in [8–13].

The pioneering work by Picasso [14] marks a starting point for time-dependent problems approached from an anisotropic context: thus, advection–diffusion-reaction problems [15] as well as purely diffusive problems [16] (to cite some relevant contributions in the field), have been investigated in recent years by using anisotropic adaptivity. In all these works, linear finite elements are employed for the spatial discretization, and the a posteriori error estimate can essentially be described as a residual-based estimator weighted by estimations of the derivatives of the solution. A more refined control of the error is achieved when one uses the solution of an adjoint problem as a weight in the computation of the error for a certain functional of the solution; this technique takes the name of 'Dual Weighted Residual' (DWR) method, and was introduced by Becker and Rannacher [17] to provide 'goal-oriented adaptivity'. The DWR method may also guarantee that the global error at the final instant of time remains below a given global tolerance. However, the DWR technique presents some extra difficulties when dealing with time-dependent problems, mainly due to the high computational cost of calculating backwards in time the dual solution of the adjoint problem, and after that, the a posteriori error estimator. Some recent works [18–20] tried to address these issues, though always in an isotropic framework; lately, Belme et al. [21] combined the goal-oriented adaptation approach with an anisotropic adaptive procedure for time-dependent problems. All these difficulties notwithstanding, more and more works on a posteriori error estimation are being devoted to goal-oriented error estimators as these produce invaluable information about quantities of engineering interest rather than those in global norms. In addition, the use of higher-order elements in anisotropic mesh adaptation is still a topic of intensive research; thus, Leicht and Hartmann [22,23] dealt with quadrilateral and hexahedral, higher-order finite elements where a goal-oriented refinement based on state-of-the-art anisotropic indicators was performed.

Convection-dominated problems, relevant to fields such as Fluid Dynamics (bubble dynamics, combustion processes, free surface flows…), may greatly benefit from anisotropic adaptation. Unfortunately, convective terms are usually problematic, and a number of methods within the finite element framework have been proposed to appropriately deal with their distinctive features. They can roughly be divided into two types: 'stabilized finite element methods' which, based on the residual of the equations, add terms to the discretized Galerkin formulation cell-by-cell (see [15] as example of this technique in an anisotropic framework); and 'Lagrangian methods', where the characteristic curves of the convection operator are employed to integrate the equations in time. A conventional implementation of the Lagrangian approach tends to deform the mesh as time goes on, causing further deterioration of the numerical solution. To overcome this drawback, the 'modified method of characteristics' or 'semi-Lagrangian scheme', computes the characteristic curves backwards in time [24,25]. We shall follow this approach to tackle the convection-dominated problems presented in this work.

The main purpose of this paper is to present an anisotropic, fully adaptive, spatial–temporal algorithm for the solution of pure convection or convection-dominated diffusion problems featuring strong anisotropies, such as exponential boundary or internal layers, in which the global error can be controlled by keeping the local or truncation error below a certain tolerance at each time step, taking advantage of the fact that convective terms accumulate the errors in the direction where the transport occurs, and cause exponential decay in the crosswind direction [26]. In the adaptive algorithm, we account for those sources of errors by means of newly proposed a posteriori error indicators, related to the semi-Lagrangian approach. The error analysis allows us to capture the directionalities of the solution, and 'encode' its anisotropic character within suitable metrics, so that we can estimate, at each time subinterval, the truncation error between the numerical solution and the exact one, in the L^2-norm. To the best of our knowledge, the proposed error indicator and its implementation in a fully adaptive, anisotropic algorithm with triangular, higher-order finite elements in semi-Lagrangian schemes has not been previously analyzed. Likewise, [27] is one of the few studies concerned with unstructured triangular meshes using anisotropic linear and quadratic finite elements in stationary problems.

In this work, we shall focus our attention on time-dependent, convection-dominated problems in a fully adaptive framework; consequently, the layout of the paper tries to replicate the steps required to implement the proposed method. Section 2 is devoted to explain how the numerical solution of a convection–diffusion problem is computed via a semi-Lagrangian scheme in a finite element framework. Section 3 is concerned with the local, a posteriori error analysis of the numerical solution, which allows us to produce local error indicators in space and time (η_s^n and η_t^n, respectively). In Section 4, we make use of the information derived from the error analysis to define the shape, orientation, and size of the elements comprising the optimal triangulation, collecting them in a metric tensor; additionally, we obtain the optimal time step size for the time integration, and present the fully adaptive, spatial–temporal algorithm (Algorithm 4), which sums up all the steps needed to tackle a convection–diffusion problem according to our method. We demonstrate the behavior of the algorithm in Section 5, by means of several numerical examples, including one 'real-world' Combustion problem in two and three dimensions. Finally, we end the paper with some conclusions and comments in Section 6.