Comput. Methods Appl. Mech. Engrg. 273 (2014) 119–142

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A locking-free and optimally convergent discontinuous-Galerkin-based extended finite element method for cracked nearly incompressible solids

Yongxing Shen [a,*], Adrian J. Lew [b]

[a] Laboratori de Càlcul Numèric, Universitat Politècnica de Catalunya (UPC BarcelonaTech), 08034 Barcelona, Spain
[b] Department of Mechanical Engineering, Stanford University, Stanford, CA 94305-4040, United States

ARTICLE INFO

Article history:
Received 24 August 2013
Received in revised form 1 January 2014
Accepted 21 January 2014
Available online 6 February 2014

Keywords:
Extended finite element method
Discontinuous Galerkin
Fracture
Lifting operator
Locking
Nearly incompressible

ABSTRACT

The extended finite element method (XFEM) is an efficient way to include discontinuities, such as a crack, into a finite element mesh. The singularity at the crack tip restricts standard finite element methods to converge with a rate of at most 1/2 for the stresses, and 1 for the displacements, with respect to the mesh size. This is true for cracks in incompressible materials as well, when any of the standard techniques to sidestep locking is adopted. To attain an optimal convergence rate of 1 for stresses and of 2 for displacements with piecewise affine elements, it is necessary to enrich the finite element space with singular basis functions. The support of these singular functions is the entire plane, but to avoid decreasing the sparsity of the stiffness matrix too much, each of them is then generally localized to a neighborhood of the crack tip by multiplying by a cutoff function or a subset of a partition-of-unity basis. For nearly incompressible materials, however, the resulting basis functions no longer contain incompressible displacement fields, and hence they either lead to locking or suboptimal convergence rates. To overcome this problem, we introduce here an XFEM with optimal convergence rate and without the problem of locking for nearly incompressible materials, i.e., it possesses an error bound that does not diverge as Poisson's ratio approaches 0.5. The method is based on a primal, or one-field formulation of a discontinuous Galerkin method that we introduced earlier. This one-field formulation is obtained through the introduction of a lifting operator, but unlike most lifting operators which map inter-element discontinuities into elementwise polynomials, ours maps such discontinuities into spaces enriched with the singular behavior of the solution. This is the key idea for the method to be simultaneously locking-free and optimally convergent.

1. Introduction

The extended finite element method (XFEM) [1] is an efficient method for incorporating cracks and other types of discontinuities in a mesh without the need for re-meshing. However, most XFEMs designed for compressible materials suffer from volumetric locking in the case of near incompressibility. This observation is not surprising. In fact, it is well known that, when the elasticity tensor becomes more and more incompressible, i.e., when Poisson's ratio ν approaches 0.5, the

* Corresponding author. Address: Carrer de Jordi Girona 1-3, Campus Nord, Edifici C2, Despatx 208, 08034 Barcelona, Spain. Tel.: +34 93 40 10893; fax: +34 93 40 11825.

E-mail addresses: yongxing.shen@upc.edu (Y. Shen), lewa@stanford.edu (A.J. Lew).

http://dx.doi.org/10.1016/j.cma.2014.01.017
0045-7825/

piecewise P_1 displacement space locks, and piecewise P_2 and P_3 elements as well as quadrilateral elements of any order may also lose their optimal order of convergence [2]. Hence any XFEM developed by enriching such basis functions locks as well.

In the context of crack problems, there are mainly two strategies to overcome the locking problem: (a) building a displacement-based XFEM by enriching a locking-free polynomial basis, and (b) resorting to a mixed method. Of these strategies, the second one applies to both near and perfect incompressibility. The stability of such methods is given by the well-known inf-sup condition which restricts the choice of displacement–pressure combinations [3]. The best known combinations include the Q2P1 (piecewise Q_2 for the displacement and piecewise P_1 for the pressure) and the Mini element [4]. However, selecting stable displacement–pressure spaces for an XFEM is not as straightforward as for a standard finite element method (FEM). A numerical study of such criterion for XFEMs was done by Legrain et al. [5], who explored a couple of stable combinations of displacement–pressure spaces (the Mini element [4] and the T6T3), which provides some general guidelines for choosing the basis functions. Nevertheless, combinations with provable stability is still absent to date to our best knowledge, perhaps because the singular enrichment functions depend on the crack tip location, and as a result, the inf-sup condition has to be analyzed for all possible topologies between the crack and the mesh. Also within the realm of mixed methods are the methods of enhanced assumed strain and reduced integration, which were used by [6,7] to study cracked nearly incompressible materials, respectively, both in the XFEM context.

Along the line of displacement-based methods, the discontinuous Galerkin (DG) method can be used as a foundation to build locking-free polynomial basis. Among the many locking-free DG methods here we mention [8–11]. In particular, we proved the locking-free property of [11] (with a minor modification) in [12].

In this paper, we build a locking-free and optimally convergent XFEM based on the locking-free DG method developed in [11] and the optimally convergent XFEM developed in [13], whose stability and optimal convergence were proved in [14]. We will enrich the DG displacement space of [11] with the mode I and mode II crack tip asymptotic solutions over a small region of the computational domain. Like in [13], we enrich a region that always contains a *fixed* ball centered at the crack tip, a condition necessary for optimal convergence rates [15,16]. Moreover, we include the nonlocal asymptotic solutions directly in the basis *without* the use of partition of unity, in contrast to most XFEMs.

The main difference between the displacement approximations here and that in [13] is that the P_1 basis functions in this work are discontinuous between all element boundaries in order to overcome locking, whereas those in [13] are discontinuous only between the two sides of the crack and between the enriched and unenriched regions. In other words, the method in [13] was not aimed for nearly incompressible materials, and will lock if used.

With the displacement approximations chosen, the proposed method is defined through the introduction of the *DG-derivative*, which plays the role of the classical derivative in the case of a continuous displacement space. The DG-derivative of a displacement approximation is the sum of two parts: (a) the displacement derivative within an element, and (b) the *lifting operator* that takes into account the inter-element discontinuities of the displacement.

The lifting operator that we will introduce later incorporates a few important considerations to ensure the simultaneous achievement of the locking-free property and optimal convergence. Firstly, when restricted to each element, the space from which we seek the lifting operator, called the *lifting space*, is that of the linear combination of (a) a polynomial tensor space, and (b) the restriction of the strain of the displacement enrichments. We remark that most DG methods to date (see the unified treatment [17] and also our previous work [11,13]) normally choose the lifting space so that it contains only polynomial functions. Secondly, in the definition of the lifting operator, we employ a similar but different space as the test space, which involves the *stress* of the displacement enrichments. We will justify the apparent complexity of this definition of the lifting operator for the consideration of absence of locking and of optimal convergence.

Compared with [14], the lifting operator in this work is not necessarily injective, a condition used by many other authors [11,18] in proving the stability and optimal consistency error of the respective DG method. In fact, we will show that this injectivity is not a necessary condition for stability or optimal consistency error; the lifting operator may not be injective but the overall method is still stable and the consistency error is optimal. This conclusion can simplify the stability proof in [14].

One important difference between the method introduced here and that in [13] is that in the method in this manuscript we cannot guarantee that the stress intensity factors (SIFs) taken as coefficients of the enrichments converge to their exact values at a rate proportional to the square root of the mesh size. The argument therein does not carry over to the method here.

The remaining sections of this paper is organized as follows. We first state the problem of interest in Section 2, then in Section 3 justify the apparent complexity of the proposed method by offering a bird's-eye view of it. In Section 4 we present the method itself in detail. Then in Section 5, we verify our method with numerical examples: one set of examples with a straight crack and one set with a circular arc crack. After these, we devote Section 6 to the analysis of the method, offering the essential ingredients of the proof of its optimal convergence and absence of locking. In particular, in Section 6.1 we prove (a) that the method is stable and (b) that the order of consistency error is optimal, both *without* invoking the injectivity of the lifting operator, a property that our proposed method may not possess. In Section 6.2 we provide a sketch of the proof for the existence of an optimal and locking-free interpolant, in the special case of no cut elements (i.e., all subelements are full elements). Finally, in Appendix A we show the less ideal performance (either suboptimal convergence or larger errors) of two (simpler) unsuccessful combinations of the lifting spaces, supporting the justification of Section 3.