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Development of a stabilised Petrov–Galerkin formulation for conservation laws in Lagrangian fast solid dynamics

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A B S T R A C T

A stabilised second order finite element methodology is presented for the numerical simulation of a mixed conservation law formulation in fast solid dynamics. The mixed formulation, where the unknowns are linear momentum, deformation gradient and total energy, can be cast in the form of a system of first order hyperbolic equations. The difficulty associated with locking effects commonly encountered in standard pure displacement formulations is addressed by treating the deformation gradient as one of the primary variables. The formulation is first discretised in space by using a stabilised Petrov–Galerkin (PG) methodology derived through the use of variational (work-conjugate) principles. The semi-discretised system of equations is then evolved in time by employing a Total Variation Diminishing Runge–Kutta (TVD-RK) time integrator. The formulation achieves optimal convergence (e.g. second order with linear interpolation) with equal orders in velocity (or displacement) and stresses, in contrast with the displacement-based approach. This paper defines a set of appropriate stabilising parameters suitable for this particular formulation, where the results obtained avoid the appearance of non-physical spurious (zero-energy) modes in the solution over a long term response. We also show that the proposed PG formulation is very similar, and under certain conditions identical, to the well known Two-step Taylor Galerkin (2TG). A series of numerical examples are presented in order to assess the performance of the proposed algorithm. The new formulation is proven to be very efficient in nearly incompressible and bending dominated scenarios.

1. Introduction

Dynamic explicit displacement-based finite element codes, based on low order finite element technology, are commonly used for the simulation of large strain impact problems by aerospace, automotive and manufacturing industries [1–6]. In these codes, the 4-noded underintegrated quadrilateral element, within the context of two dimensions, is the preferred option to model solid components, avoiding detrimental volumetric locking under nearly incompressible dynamic behaviour. However, many practical applications, such as crashworthiness and drop-impact modelling, involve geometries that are far too complex to be meshed using quadrilaterals in two dimensions (or hexahedra in three dimensions) [7–10].

The presence of large solid deformations accompanied by severe mesh distortion may lead to poorly shaped elements unless some form of adaptive remeshing is also applied [11]. At present, the possibilities of using mesh adaptation methods with hexahedral elements for explicit Lagrangian dynamics are very limited. A number of powerful unstructured triangular (or tetrahedral) mesh generators and related mesh adaptivity procedures are available for Computational Fluid Dynamics

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(CFD) applications [12,13]. Unfortunately, the standard (or displacement-based) formulation for linear triangular elements is known to exhibit volumetric locking for nearly incompressible deformations.

It is also known that the use of linear interpolation in the displacement-based Finite Element Method (FEM) leads to second order convergence for the primary variables (i.e. displacements) but one order less for derived variables (i.e. strains and stresses) [14], requiring some form of stress improvement procedure if the latter are of interest [15,16].

From the time discretisation point of view, the Newmark method has a tendency to introduce high frequency noise in the solution field, especially in the vicinity of sharp spatial gradients. Perhaps more importantly, its accuracy is remarkably degraded once numerical artificial damping is employed [17]. In [18–21] some minor modifications have been introduced to improve the effect of the numerical dissipation without including a discontinuity sensor. However, such schemes are then unsuitable for shock dominated problems.

A wealth of numerical techniques are available in the literature to deal with locking phenomena. A non-optimum approach relies upon the constant refinement of the mesh size (i.e. h-refinement), which can alleviate the shear locking phenomenon in the FEM but has no ability to resist volumetric locking in the nearly incompressible regime [22]. Alternatively, the flexibility of the formulation can be enhanced by introducing the so-called p-method, in which the polynomial order within elements is increased on a fixed mesh [23–26]. However, there is evidence that the accuracy of the solution at any fixed polynomial order is far from optimal with relatively coarse meshes using the standard displacement-based formulation [27].

A general approach to handle incompressibility is to introduce a multi-field Fraeijs de Veubeke–Hu–Washizu (FdVHW) type variational principle [28], which enables the use of independent kinematic descriptions for the volumetric and deviatoric components of the deformation. In particular, the mean dilatational formulation [29], in which the lowest possible (i.e. constant) interpolation is employed for the volumetric component of the deformation, is widely used in commercial codes [1,2]. This specific technique can be identified as a particular case of Selective Reduced Integration (SRI), where the volumetric deformation is suitably underintegrated.

Unfortunately, the mean dilatational formulation cannot be employed in classical linear FEM for two dimensional triangles or three dimensional tetrahedrons, as these elements already make use of the simplest Gaussian quadrature rule (i.e. one point) for exact integration of the stiffness matrix. Therefore, it is imperative to use some form of 'projection' to reduce the number of volumetric constraints [29–34]. Reference [9] suggests that the volumetric strain energy functional can be approximated by evaluating averaged nodal pressures in terms of tributary nodal volumes whilst the deviatoric component is treated in a standard manner. However, the resulting solution was reported to behave poorly in bending dominated simulations [35]. To circumvent this difficulty, reference [36] proposes a new linear tetrahedron (linear triangle in two dimensions) by applying a nodal averaging process to the whole small strain tensor. Reference [37] extends this application to the large strain regime with the idea of employing an averaged nodal deformation gradient tensor as the main kinematic variable. As reported in [11,33,34], the resulting formulation suffers from artificial mechanisms similar to hourglassing.

As is well known for incompressible mixed finite element formulations, the order of approximation for both displacement (or velocity) and pressure variables cannot be chosen arbitrarily. The resulting scheme has to satisfy the Ladyzenskaya–Babuska–Brezzi (LBB) condition [23] to ensure stability and optimal convergence. References [17,31] report that this mixed formulation has a close link with the $\overline{B}$ strain projection technique in the small strain regime. Generalisation of $\overline{B}$-type methods to hyperelastic finite deformations, namely $\overline{F}$-type methods, have been proposed but still require further research [38,39].

The main goal of this paper is the introduction of a stabilised Petrov–Galerkin (PG) [40] framework within the context of fast Lagrangian dynamics, by using equal low order approximation for all variables, aiming to resolve the shortcomings mentioned above for the traditional formulation. The numerical algorithm is derived from a variational framework through the use of stabilised work-conjugate principles, without having to utilise the flux Jacobian matrix in the formulation. The resulting formulation is analogous to the Variational Multi-Scale (VMS) method [41–44] where a series of time rate and time integrated residual based terms will be employed for stabilisation.

A mixed methodology is presented in the form of a system of first order conservation laws, where the linear momentum, the deformation gradient tensor and the total energy of the system are regarded as the three main conservation variables of this mixed approach [45]. Insofar as both the linear momentum and the deformation gradient tensor are primary variables of the problem, the stresses converge at the same rate as the velocities (or displacements) which proves ideal in the case of low order elements. Moreover, the new formulation seems to be very efficient in nearly incompressible and bending dominated scenarios.

This paper is organised as follows. Section 2 introduces a mixed conservation law formulation for fast structural dynamics, where the linear momentum, the deformation gradient tensor and the total energy are treated as primary conservation variables, within the context of large deformations. Section 3 describes the development of the PG stabilised spatial semi-discretisation, preventing the appearance of numerical instabilities in solving the convection-dominated problem. It is also demonstrated that the PG formulation is identical to the Two-step Taylor Galerkin (2TG) methodology, provided certain conditions are met. In Section 4, the use of an explicit Total Variation Diminishing Runge–Kutta (TVD-RK) time marching scheme is described. Section 5 summarises the solution algorithm of a stabilised PG formulation in conjunction with the TVD-RK time integrator for implementation purposes. In Section 6, a series of numerical examples are compared with some of the existing techniques to demonstrate the robustness of the proposed numerical methodology. Finally, Section 7 presents some concluding remarks and current directions of research.