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Anisotropic mesh adaptation with optimal convergence for finite elements using embedded geometries

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A B S T R A C T

This paper presents a numerical study of a recent technique that consists in modeling embedded geometries by a level-set representation in combination with local anisotropic mesh refinement. The local anisotropic mesh procedure is suitable for various orders *p* of finite element approximations. This method proves beneficial in simulations involving complex geometries, as it suppresses the need for the tedious process of body-fitted mesh generation, without altering the finite element formulation nor the prescription of boundary conditions. The first part of the study deals with a simple Laplace problem featuring a planar interface on which a Dirichlet boundary condition is imposed. It is shown that the appropriate amount of local isotropic refinement yields the optimal convergence rate for various finite element orders *p*, unlike uniform refinement. Anisotropic refinement further ensures geometric convergence and limits the growth of the number of unknowns. Then, we explain how to use metric-based anisotropic adaptation to obtain *nearly* body-fitted meshes with arbitrary geometries. The optimal rate of convergence, both for the solution and the geometry, is demonstrated on 2D and 3D academic Laplace problems involving curved boundaries. Finally, applications in the field of fluid dynamics and material science are presented. The results for these simulations successfully converge towards data from the literature, analytical solutions or values obtained with body-fitted meshes.

1. Introduction

In the context of computational fluid and solid mechanics, numerical methods are applied to increasingly challenging problems that often involve complex geometries. The usual strategy to handle such cases is to firstly generate an unstructured mesh which conforms to the geometry, and then solve the physical problem with a numerical method that works with unstructured meshes, such as the Finite Element Method (FEM). However, most meshing software packages rely on a high-quality CAD description of the domain geometry, which is not readily available in the traditional workflow of professionals in sectors like architecture [11]. In the field of medicine, the quality of the primary CAD models obtained through imaging techniques may not be sufficient to allow direct meshing [24]. If the geometry evolves in time, the mesh is usually modified during the simulation, either by deformation techniques relying on elastic analogies, which sometimes lacks robustness, or by complete remeshing, which is computationally expensive.

Similar issues, that arise in multi-physics problems involving moving interfaces, have fostered research efforts to embed the description of the geometry in the formulation of the numerical method. In this way, the meshing work needed to take

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into account complex features such as cracks and material interfaces can be dramatically reduced. Moreover, this removes the need for high-quality CAD data, as no topological information on the geometry is then required in the meshing process.

In the immersed boundary method [30], fluid–solid interfaces are modeled from a Lagrangian point of view as a set of discrete force generators. Other approaches retain an Eulerian description consistent with the underlying numerical method by relying on the advection of interface-tracking quantities. Among them, the Marker-and-Cell [27] and the Volume of Fluid methods [1] make use of "marker" quantities that require a specific numerical treatment because of their discontinuous character. A more convenient continuous representation of the interface can be obtained through level-set functions [7,26]. Level-set techniques have also been used recently to model fixed, but complex boundaries in an immersed volume framework [20].

While level-set curves and other types of interface-tracking functions are an efficient solution for representing geometries embedded in non-fitted meshes, they make it more difficult to impose boundary conditions. Indeed, they prevent nodal collocation, which makes the prescription of Dirichlet conditions particularly challenging. To deal with this problem, some approaches based on penalty methods or Lagrange multipliers have been proposed in the literature [13,21]. The issue of Babuška-Brezzi stability has been addressed using either stable [4] or stabilized [3,14] approaches, that have drawbacks. The stable method presented in [4] is relatively complex, as it requires the definition of a specific set of Lagrange multipliers that depend on the topology of the mesh. Stabilized methods are more standard with respect to the finite element technology, but choosing appropriate values for the parameters that control the stabilization may not be obvious. More importantly, both approaches are intrusive in the sense that they require deep modifications in finite element kernels, either by introducing new finite element unknowns [29] or by modifying standard finite element formulations (or both).

In this paper, we follow another approach that relies on local anisotropic mesh adaption [7,2,12] around the geometry to obtain *nearly* body-fitted meshes, which combines the benefits of the aforementioned methods:

1. The Dirichlet boundary condition is imposed in a strong manner by nodal collocation, just as with body-fitted meshes.
2. The standard finite element formulation is used as is, without resorting to basis enrichment or Lagrange multipliers that alter its numerical properties.
3. The mesh does not conform exactly to the embedded geometry, which facilitates the meshing process and makes it possible to use level-set functions instead of high-quality CAD data.

Because anisotropic mesh adaption procedures have gained in robustness and availability in the last decade, this approach has become competitive with respect to the modification of finite element formulations. The aim of this paper is to show that optimal convergence rates can be easily obtained using the standard finite element discretization of the variational weak form, with barely the same number of degrees of freedom as in a conforming mesh approach and without changing the finite element formulation. Both two- and three-dimensional examples illustrate the new approach.

The paper is organized as follows. Section 2 gives an overview of the adaptive strategy. Section 3 involves two-dimensional examples while numerical results for three-dimensional cases are discussed in Section 4. We show applications of this approach in Section 5 and draw conclusions in Section 6.

2. Optimal *nearly* body-fitted meshes

Our technique for imposing Dirichlet boundary conditions on embedded surfaces relies on the generation of an anisotropic adaptive mesh for which the mesh size is carefully chosen in order to ensure the optimal convergence of both the solution and the geometry of the interface.

2.1. General principle

Let M be a mesh of a domain Ω that is composed of n_e elements e_i, $i = 1, \ldots, n_e$ and n_v vertices $\mathbf{x}_i$, $i = 1, \ldots, n_v$. Consider an embedded interface Γ (see Fig. 1) that can be modeled by the iso-zero value of a level-set function $\phi(\mathbf{x})$. The level-set function is then defined in all the domain as the signed distance to the interface.

In practice, the level-set function $\phi(\mathbf{x})$ is evaluated at the center of gravity $\mathbf{c}_i$ of every element e_i. The sign of $\phi(\mathbf{c}_i)$ determines whether element e_i is either on one side or on the other side of the levelset. In Fig. 1, colored elements correspond to $\phi(\mathbf{c}_i) > 0$ and non-colored elements correspond to $\phi(\mathbf{c}_i) < 0$. Mesh edges that separate colored and non colored elements constitute the discrete approximation Γ^* of the continuous interface Γ.

This way of treating the interface has obvious advantages. No enrichment is needed to account for intra-element features, as it is the case in X-FEM. Standard finite element formulations can therefore be used as is for solving a problem with an embedded interface. Yet, it is well known that this treatment of interfaces leads to a poor first order of convergence in finite element simulations [28]. In this paper, we address this issue using anisotropic mesh adaptation near the interface Γ. The fact that mesh generators and finite element solvers are independent software components makes this approach appealing in practice.