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Bounding the influence of domain parameterization and knot spacing on numerical stability in Isogeometric Analysis

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ABSTRACT

Isogeometric Analysis (IGA) was introduced by Hughes et al. (2005) [1] as a new method to bridge the gap between the geometry description and numerical analysis. Similar to the finite element approach, the IGA concept to solve a partial differential equation leads to a (linear) system of equations. The condition number of the coefficient matrix is a crucial factor for the stability of the system. It depends strongly on the domain parameterization, which provides the isogeometric discretization. In this paper we derive a bound for the condition number of the stiffness matrix of the Poisson equation. In particular, we investigate the influence of the domain parameterization and the knot spacing on the stability of the numerical system. The factors appearing in our bound reflect the stability properties of a given domain parameterization.

1. Introduction

The concept of Isogeometric Analysis (IGA) was first proposed by Hughes et al. [1] in 2005 to provide a seamless integration of Computer-Aided Design (CAD) and Finite Element Analysis (FEA). In IGA, the same basis functions are used for the geometry description and for the numerical analysis. One major advantage of IGA over the classical finite element method (FEM) is the improved representation of the computational domain by using non-uniform rational B-splines (NURBS) or other classes of basis functions. Furthermore, due to the increased smoothness of the basis functions, the numerical solution inherits a high continuity.

Since the first publication of Hughes and his co-workers on IGA, an increasing number of researchers worldwide are working in that field, applying the new methodology to a wide variety of simulation problems. Within the last 8 years, 184 papers with the word "isogeometric" in the title have been published alone in the journal Computer Methods in Applied Mechanics and Engineering. The number of publications per year is increasing, which shows that IGA is a very active field of research.

Besides practical issues, also the theoretical foundations of IGA have been analyzed thoroughly. Here we mention a few representative results: Fundamental results on approximation properties, error estimates and numerical stability in IGA are described in [2–4]. Among other issues, these publications derive error estimates for approximation by NURBS functions with respect to degree, smoothness and stepsize (knot spacing). Another important issue is the derivation and analysis of efficient quadrature rules for IGA, see [5].

Especially when dealing with large problems, the computational effort for solving the numerical system becomes an important issue. It is then preferable to use iterative solvers instead of direct solvers to reduce the computational costs. When using an iterative solver, the condition number is a crucial factor, since it highly influences the rate of convergence.

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Furthermore, small changes in the right-hand side of a linear system could cause big changes in the solution, if the system matrix is not well conditioned.

In the classical FEM literature one can find considerations about the stiffness matrix and bounds for the condition number, being of order $\mathcal{O}(h^{-2})$ for uniform discretizations with grid size h, see, e.g., [6–8]. This also generalizes to locally refined meshes under mild restrictions on the refinement, see, e.g., [9]. There are also recent publications giving finer bounds for p-FEM, see, e.g., [10]. Moreover, it has been analyzed how the quality of the underlying mesh influences the numerical properties of the stiffness matrix, see, e.g., [11] and the references cited therein.

Many results from classical FEM can be carried over to the isogeometric approach. However, when we want to compute bounds for the condition number of the stiffness matrix, some differences occur due to the presence of the geometry mapping and the larger support of the basis functions.

The condition number in IGA depends on the underlying parameterization of the computational domain. Hence, this number can be used to measure the quality of a parameterization. Our goal is to analyze the influence of the parameterization of the domain of interest on the stability of the numerical system. More precisely, our aim is to derive an upper bound for the condition number, which reflects the quality of the domain parameterization.

Construction of domain parameterizations that are suitable for IGA have been presented in several publications. In [12] we provided a tool to construct B-spline or NURBS swept volumes via a variational framework. This class of volumes is suitable to parameterize functional free-form shapes such as blades for turbines and propellers. Moreover, we discussed the influence of the chosen parameterization on the accuracy of the solution. However, we had no simple measure for the quality of a parameterization.

Cohen et al. [13] introduce the framework of analysis-aware modeling, where model properties and parameters should be selected to facilitate Isogeometric Analysis. Martin et al. [14] provide a method to construct volumetric B-spline parameterizations from input genus-0 triangle meshes using harmonic functions.

In [15–17] the authors show that the parameterization of the computational domain has an impact on the simulation result and the efficiency of the computations. An optimal parameterization of the computational domain is generated by a shape optimization method. Additionally, an easy-to-check algorithm to ensure that the constructed parameterization has no self-intersections is proposed.

The articles [18,19] analyze the influence of singularities in the parameterization of the physical domain. The authors present regularity results and modification schemes for the test function space in the case of reduced regularity. In [20] a method for shape optimization using the Winslow functional is introduced. In [21,22] the authors use this technique to optimize the domain of interest for special applications such as vibrating membranes or conducting scatterers.

In [23–26] the authors discuss various methods for shape optimization of different domains. Lipton et al. [27] discovered the effect of severe distortion of the control and physical mesh. In all cases, using higher order basis functions leads to increased robustness under mesh distortion.

In many of these earlier studies, a parameterization was considered to be "good" in the sense that it is analysis-suitable, whenever the physical mesh looks "nice". The ratio of maximum to minimum physical element size should not exceed prescribed limits and the physical elements should not be too much distorted. However, this was a rather heuristic way to define a "good" parameterization. As far as we know, there are no publications considering the influence of the parameterization on the stability of the numerical system.

Recently, Gahalaut and Tomar [28] derived estimates for the condition number of the stiffness and mass matrix for IGA for h- and p-refinement. Their estimates depend on the polynomial degree of basis functions p and the mesh size h and some constant. The dependence on the geometry mapping G, however, is hidden in the constant.

In contrast to this approach, in our paper we derive bounds for the condition number of the stiffness matrix which make explicit the influence of the domain parameterization. The investigated bounds depend on the knot vector and on a term arising from the geometry mapping. Special attention will be paid to the latter term, since this provides a quality measure for the parameterization.

The remainder of this paper is organized as follows: In Section 2 we recall the principles of IGA and introduce our notation. Section 3 describes the outline and the basic steps of our approach to find a bound for the condition number of the stiffness matrix. In the following three sections we will derive bounds for the one-, two- and three-dimensional case. We will present several examples and compare different parameterizations in Section 7. Finally we will conclude the paper in Section 8.

2. Preliminaries

We recall the principles of IGA and formulate the model problem. Furthermore, we recall some elementary results about quadratic forms and matrix inequalities.

2.1. IGA on a single patch

Consider the unit cube $\Omega_0 = [0,1]^D$ in $\mathbb{R}^D$, where $D \in \{1,2,3\}$ is the dimension. The cube Ω_0 is called the parameter domain. We denote by $\{\beta_i^d\}_{i=1,\dots,n_d}$ the univariate B-splines of degree p_d for the dth parameter direction, which are defined by a knot vector