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Variational gradient plasticity at finite strains. Part II: Local–global updates and mixed finite elements for additive plasticity in the logarithmic strain space

Christian Miehe *, Fabian Welschinger, Fadi Aldakheel

Institute for Applied Mechanics (CE), Chair I, University of Stuttgart, Pfaffenwaldring 7, 70569 Stuttgart, Germany

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ABSTRACT

The second part of our work on variational inelasticity with long-range effects outlines the formulation and finite element implementation of additive finite gradient plasticity in the logarithmic strain space. It is considered to be the most simple approach to finite plasticity, suitable for the *purely phenomenological description of polycrystalline metals or amorphous materials*, if structures of the geometrically linear theories are defined in the Lagrangian logarithmic strain space. We start from a *mixed saddle point principle* for metric-type additive plasticity, which is specified for the important model problem of isochoric von Mises plasticity with gradient-extended hardening/softening response. The mixed variational structure includes the hardening/softening variable itself as well as its dual driving force. The numerical implementation exploits the underlying variational structure, yielding a canonical symmetric structure of the monolithic problem. It results in a novel finite element design of the coupled problem incorporating a long-range hardening/softening parameter and its dual driving force. This allows a straightforward *local definition* of plastic loading–unloading driven by the long-range fields, providing very *robust* finite element implementations of gradient plasticity. It includes a rational method for the definition of elastic–plastic-boundaries (EPBs) in gradient plasticity along with a postprocessor that defines the plastic variables in the elastic range. We discuss alternative mixed finite element designs of the coupled problem, including a local–global solution strategy of short- and long-range fields. All methods are derived in a rigorous format from variational principles. Numerical benchmarks demonstrated the excellent performance of the proposed mixed variational approach to gradient plasticity in the logarithmic strain space.

1. Introduction

Several approaches to the kinematic formulation of *purely phenomenological* finite plasticity are established in the literature. We refer to the classical works Green and Naghdi [1], Lee [2], Rice [3] and Mandel [4] on this field. The micromechanically-based continuum slip theory for single crystals includes a *multiplicative definition* of an objective elastic strain measure, such as $\boldsymbol{\varepsilon}^e := \ln[\boldsymbol{F}^{p-T}\boldsymbol{C}\boldsymbol{F}^{p-1}]/2$ in terms of a *plastic map* $\boldsymbol{F}^p \in GL_+(3)$ that includes a plastic rotation. The plastic map represents the non-material flow of dislocations through a material lattice, see for example Rice [3], Kröner and Teodosiu [5] and Mandel [4]. The plastic rotational part is well-defined in crystal plasticity but is often controversially discussed, when multiplicative plasticity is used for the purely phenomenological modeling of polycrystals, polymers or geomaterials, see e.g. Naghdi

* Corresponding author. Tel.: +49 711 685 66379.
E-mail address: christian.miehe@mechbau.uni-stuttgart.de (C. Miehe).
URL: http://www.mechbau.uni-stuttgart.de/ls1/ (C. Miehe).

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[6]. Phenomenological assumptions for the plastic rotation often appear somewhat artificial, for example by simply setting the plastic spin to zero as considered in Moran et al. [7], Miehe [8] and Simó [9]. This has motivated to formulate a framework of finite plasticity based on a *plastic metric* $\boldsymbol{G}^p \in Sym_+(3)$, see Miehe [10,11], that is considered as a primitive internal variable in the sense of Green and Naghdi [1] and Papadopoulos and Lu [12,13]. It was shown in Miehe et al. [14], that a framework of metric plasticity based on the specific *additive Lagrangian form* $\boldsymbol{\varepsilon}^e := \ln[\boldsymbol{C}]/2 - \ln[\boldsymbol{G}^p]/2$ of an objective elastic strain measure is close to models of (plastic-spin-free) multiplicative plasticity, for both the isotropic J_2-theory of von Mises as well as anisotropic Hill-type plastic flow. As a consequence, such a framework is well suited for the *purely phenomenological modeling of finite plasticity*, where the plastic flow is material in nature, causing a plastic deformation of structural directors. As shown in Miehe et al. [14,15], the key advantage of this approach is its *modular structure*, that opens the possibility to place well-known constitutive structures of geometrically linear plasticity into the logarithmic strain space, surrounded by a *model-inherent tensorial pre- and postprocessing*, which is purely geometric in nature. This is in contrast to the earlier works Eterovic and Bathe [16], Perić et al. [17], Simó [18,9], and Cuitino and Ortiz [19] on isotropic multiplicative finite plasticity with linear elastic models in the logarithmic elastic strain space, where the geometric pre- and postprocessing appeared only as a part of the update algorithm. Hence, additive metric-type plasticity in the logarithmic strain space offers a consistent and straightforward approach for the extension of geometrically linear constitutive structures to finite strains. Following the first part Miehe [20] of our work on finite inelasticity with length scales, the goal of this paper is to outline a computational framework of *gradient plasticity in the logarithmic strain space* based on a rigorous exploitation of mixed variational principles.

Several observations underline the need for non-standard continuum approaches to finite plasticity. A first physically-based motivation is the experimentally observed *increase in strength* of metallic structures with diminishing size, resulting from dislocation related hardening effects, see for example Hall [21], Petch [22], Fleck et al. [23] and Arzt [24]. A further key motivation for the use of gradient plasticity arises from the computation of localized plastic deformation in softening materials with *finite element techniques*, yielding for local theories the pathological mesh dependencies for zero length scale. To overcome this non-physical behavior, gradient-enhanced plasticity models are used as *regularization methods*, which provide the existence of a length scale, see for example Lasry and Belytschko [25], De Borst and Mühlhaus [26], Liebe and Steinmann [27] and Engelen et al. [28]. The incorporation of such length scales is also requested by the *mathematical theory for the existence of solutions in finite plasticity*, see for example Mielke and Müller [29]. *Phenomenological theories* of gradient plasticity are outlined in the works of Aifantis [30], Mühlhaus and Aifantis [31], Gurtin [32], Forest and Sievert [33], Gudmundson [34], Gurtin and Anand [35], Reddy et al. [36] and Fleck and Willis [37,38]. In most of these works variational principles are missing, which recast the formulation of the boundary value problem of gradient plasticity in a canonical format. An example of a particular two-field rate-type variational principle was given in the pioneering works of Mühlhaus and Aifantis [31] and De Borst and Mühlhaus [26]. However, a rigorous treatment of *mixed variational principles* for gradient plasticity including driving force variables is not established in the theoretical and computational literature.

In contrast to classical local theories, where the short-range internal variables are usually approximated by piecewise discontinuous functions and condensed out at the integration points of finite elements, the key difficulty in *computational gradient plasticity* is to realize the continuity of the long-range variables. In the works of Liebe and Steinmann [27] and Liebe et al. [39], a global active set strategy of gradient plasticity was considered, where Kuhn-Tucker-type loading/unloading conditions were checked in weak form via finite element residuals at the nodes. Such a formulation needs a non-standard global active-set search, which is not robust when applied to complex inhomogeneous response.

McBride and Reddy [40] outlined a discontinuous Galerkin formulation of isotropic multiplicative gradient plasticity. The most severe inconvenience resides in the fact that the gradient-enhanced consistency condition, which is solved as an additional PDE, is only valid within the plastic domain, and no direct long-range interaction into the elastic region exists. Therefore, the evolving *elastic–plastic boundary* (EPB) depends on the solution, which complicates the numerical calculation considerably. As a consequence, spurious oscillations of the plastic variables are observed near the EPB when straightforward finite element discretizations of the full domain are applied, as reported for example in De Borst and Pamin [41], Liebe and Steinmann [27] and Liebe et al. [39]. This has motivated Engelen et al. [28], Geers et al. [42] and Geers [43] to propose a gradient plasticity model based on an accompanying PDE of the modified Helmholtz type, that defines the long-range equivalent plastic strain in terms of its short-range counterpart. Here, the starting point was a fully non-local integral-based ansatz, inspired by earlier work of Peerlings et al. [44] on gradient damage mechanics. The advantage of this setting, coined by the authors as 'implicit gradient theory of plasticity', is that this accompanying differential equation is valid in the full domain and not restricted to an EPB. However, such an approach is not variational and contradicts classical ideas of plasticity by defining an evolution of plastic variables outside of the plastic zone, reflecting the non-local nature of the method. Hence, it is not straightforward to base it on classical thermodynamical arguments. A thermodynamic foundation was given later by Peerlings et al. [45] and Forest [46] in the context of a micromorphic theory. Though such a framework is probably the most simple and easy to implement approach for an ad hoc extension of classical local plasticity towards a theory accounting for size effects, we put our attention to the classical settings of gradient-regularized plasticity reviewed above. The key motivation is to construct a new type of numerical implementation of von Mises-type gradient plasticity that is *variational in nature, robust and straightforward to implement*, and therefore competitive with the above mentioned 'implicit gradient theories'.

The aim of this work is to outline a new theoretical and computational setting for gradient plasticity in terms of a rigorous use of *mixed variational principles for the evolution problem*. In line with the general framework outlined in the first part of this work Miehe [20], we derive consistent mixed variational principles for the evolution problem of von Mises-type finite