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On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime

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ABSTRACT

According to the Helmholtz decomposition, the irrotational parts of the momentum balance equations of the incompressible Navier–Stokes equations are balanced by the pressure gradient. Unfortunately, nearly all mixed methods for incompressible flows violate this fundamental property, resulting in the well-known numerical instability of poor mass conservation. The origin of this problem is the lack of L^2-orthogonality between discretely divergence-free velocities and irrotational vector fields. Therefore, a new variational crime for the nonconforming Crouzeix–Raviart element is proposed, where divergence-free, lowest-order Raviart–Thomas velocity reconstructions reestablish L^2-orthogonality. This approach allows to construct a cheap flow discretization for general 2d and 3d simplex meshes that possesses the same advantageous robustness properties like divergence-free flow solvers. In the Stokes case, optimal a priori error estimates for the velocity gradients and the pressure are derived. Moreover, the discrete velocity is independent of the continuous pressure. Several detailed linear and nonlinear numerical examples illustrate the theoretical findings.

1. Introduction

In the last forty years mixed finite elements for the incompressible Navier–Stokes equations have seen a great success in mathematical fluid dynamics [30,8,44,18,34,16]. The theory of mixed finite elements is elegant and compact, and it delivers rather simple recipes for the construction of convergent numerical schemes with easily predictable convergence rates and other distinctive properties. Obviously, the great flexibility of mixed finite elements is mainly indebted to the relaxation of the divergence constraint [8,31]. However, there is a price to pay for this relaxation. This price can be observed most easily from the typical a priori mixed finite element estimate for the incompressible Stokes equations

$$\begin{aligned} &-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f}, \quad \mathbf{x}\in\Omega,\\ &-\nabla\cdot\mathbf{u}=0, \quad \mathbf{x}\in\Omega,\\ &\mathbf{u}=\mathbf{0}, \quad \mathbf{x}\in\partial\Omega \end{aligned} \tag{1}$$

that reads as

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\leqslant C_1h^k\,|\,\mathbf{u}|_{k+1}+\frac{C_2}{\nu}h^k\,|\,p|_k, \tag{2}$$

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employing finite elements of polynomial order k for the discrete velocities and finite elements of order $k-1$ for the discrete pressure [8,30,44]. Here, the price consists in the error term $\frac{C_2}{\nu} h^k \mid p|_k$, which is classical for *exterior*, i.e., non-divergence-free, mixed methods in the sense of Ref. [30], but links the discrete velocity $\mathbf{u}_h$ in a disadvantageous manner with the continuous pressure p. This price, which by the way can be avoided completely by divergence-free mixed methods like the classical Scott–Vogelius element [50,49,42,52] or more recent divergence-free discretization approaches [11,12,51,20,21,32,22], is well-known in the finite element community as poor mass conservation [37,36]. In mixed finite elements it is traditionally tried to get around with poor mass conservation by stabilization techniques like grad-div stabilization [23,44,41,40,35,27,5,10,39] or by variable transformations reducing the complexity of the continuous pressure [26,48]. An interesting, alternative approach in the Stokes case was proposed in [28], employing a (discrete) Helmholtz decomposition of the exterior forcing. But these techniques always seem to mitigate this problem only, are restricted to special cases and never solve it completely [25,33]. But although poor mass conservation has accompanied the development of mixed methods for incompressible flows for several decades, surprisingly only in recent years research on poor mass conservation began to receive a broader attention and a better understanding of it started [41,40,9,36,37,25,10,38].

Nowadays, it is clear that poor mass conservation is the reason for several different kinds of non-physical behaviour that is shown by (exterior) mixed methods for incompressible flows. As a nice example, the observations by Dorok et al. are mentioned here, which are already several years old [17]. There, the authors show that in a heated cavity a change of the absolute values of some inhomogeneous Dirichlet boundary conditions for the temperature changes the discrete velocity field, although in the continuous problem only the difference between the different temperatures matters. The reason for this strange behavior of *exterior* mixed methods roots in a (discrete) violation of a fundamental invariance property of the (continuous) incompressible Navier–Stokes equations (with homogeneous Dirichlet boundary conditions). Changing the exterior forcing by $\mathbf{f} \to \mathbf{f} + \nabla\psi$, changes the Navier–Stokes solution by $(\mathbf{u}, p) \to (\mathbf{u}, p + \psi)$, i.e., the velocity does not change and the additional forcing is balanced by the pressure gradient. In Dorok et al. [17], it is nicely explained that in their heated cavity problem a change of the absolute temperature level induces an additional irrotational buoyancy term that is is treated in a non-physical way by exterior mixed methods. Continuing this discussion, one should note that there is an important special case of the fundamental invariance property of incompressible flows above, where exterior mixed methods can collapse dramatically. This special case occurs, whenever the forcing $\mathbf{f}$ is completely irrotational, i.e., $\mathbf{f} \equiv \nabla\psi$. In such cases, the forcing does not excite any motion in a fluid, but exterior mixed methods usually suffer from large spurious velocity oscillations, which have been observed in the past by several works [28,17,24,36,10]. Another example of non-physical poor mass conservation is visible, when the Coriolis force is added to the incompressible Navier–Stokes equations. Since the Coriolis force $2\Omega \times \mathbf{u}$ may have a large irrotational part in the sense of the Helmholtz decomposition in some physical situations (e.g., in two dimensions, the Coriolis force is always irrotational [14]!), it excites spurious velocity oscillations [14,13], since discretely divergence-free velocity fields in exterior mixed methods are not orthogonal to this irrotational forcing in the L^2 scalar product. A very similar, well-known example is the rotational form of the incompressible Navier–Stokes equations [35], where the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is replaced by $\omega \times \mathbf{u}$ with $\omega = \nabla \times \mathbf{u}$. Here, exterior mixed methods deliver different, usually remarkably worse discrete velocity fields compared to discretizations employing the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$, although the corresponding continuous velocity fields are the same [35,4]. In the numerical section of this contribution, it will be shown how an inappropriate treatment of the (usually large) irrotational part of the nonlinear term $\omega \times \mathbf{u}$ is responsible for poor accuray in simulations.

Therefore, it has recently been pointed out that poor mass conservation is maybe a misleading term, and could be described probably better by a kind of poor momentum balance, resulting from a lack of L^2-orthogonality between irrotational and discretely-divergence-free vector fields [38]. Following these ideas, in this contribution the significance of the *continuous* Helmholtz decomposition in discretizations of incompressible fluids is emphasized, and a new kind of variational crime is introduced, in order to cure the lack of L^2-orthogonality between irrotational and discretely divergence-free vector fields. Thereby, a simple, cheap and robust lowest-order discretization for the incompressible Navier–Stokes will be constructed on regular simplex grids in two and three space dimensions, which is based on the classical nonconforming Crouzeix–Raviart element [15,3,1]. Here, the variational crime consists in replacing discretely divergence-free vector fields by divergence-free lowest-order Raviart–Thomas [43,8,19] velocity reconstructions, wherever L^2 scalar products occur in the momentum balance equations. The motivation behind this discretization approach will be explained in detail in the next section. Last but not least, it should be noted that the up-to-now rare use of divergence-free Navier–Stokes discretizations in practice is probably mainly indebted to their expensive costs even on coarse meshes and too severe constraints on the used triangulations. Therefore, the variational crime proposed in this contribution could be a viable resort out of this dilemma.

The contribution is organized as follows. The new finite element method based on Raviart–Thomas velocity reconstructions for the nonconforming Crouzeix–Raviart element will be presented in Section 3. In Section 4, a priori error estimates for the incompressible Stokes equations are derived. Especially, the estimates for the discrete velocity are completely independent of the continuous pressure, like in divergence-free methods [36]. In Section 5, detailed numerical examples show the robustness and acuracy of the proposed scheme. Here, the emphasis is on avoiding poor mass conservation in the *left hand side* of the momentum balance equations, which is the harder problem to solve. Especially, it will be demonstrated that the rotational form of the incompressible Navier–Stokes equations can be safely used, if appropriate velocity reconstructions are applied.